MATHEMATICAL ASSOCIATION OF NIGERIA
(M.A.N)

abacus

THE JOURNAL OF THE MATHEMATICAL ASSOCIATION OF NIGERIA

MATHEMATICS SCIENCE SERIES
VOLUME 50, NUMBER 1,
APRIL, 2023

NR-ISSN 0001 3099

Editor-in-Chief:
Professor Muhammad Lawan Kaurangini, FICA
Aliko Dangote University of Science and Technology, Wudil, Nigeria

MATHEMATICAL ASSOCIATION OF NIGERIA
(M.A.N)

abacus

THE JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF NIGERIA

MATHEMATICS SCIENCE SERIES
VOLUME 50, NUMBER 1,
APRIL, 2023

NR-ISSN 0001 3099

Editor-in-Chief:
Professor Muhammad Lawan Kaurangini, FICA
Kano University of Science and Technology, Wudil, Nigeria

National Executive Officers
(2024-2025)

Abacus:

Submission of the paper
Authors are requested to visit www.man-nigeria.org.ng to submit manuscripts in Microsoft word and make payment accordingly.

Preparation of the manuscript
General: The manuscripts should be in English and typed with single column and single line spacing on single side of A4 paper. The first page of an article should contain;
(1) a title of paper which well reflects the contents of the paper (Arial, 12pt),
(2) all the name(s) and affiliations(s) of authors(s) (Arial, 12pt),
(3) an abstract of 100-250 words (Times New Roman, 11 pt),
(4) 3-5 keywords following the abstract, and
(5) footnote (personal title and email address of the corresponding author). The paper should be concluded by proper conclusions which reflect the findings in the paper. The normal length of the paper should be between 10 to15 journal pages.

Tables and figures: Tables and figures should be consecutively numbered and have short titles. They should be referred to in the text as following examples (e.g., Fig. 1(a), Figs. 1 and 2, Figs. 1(a)-(d) / Table 1, Tables 1-2, etc). Tables should have borders (1/2pt plane line) with the captions right before the table. Figures should be properly located in the text as an editable image file (.jpg) with captions on the lower cell.

Units and Mathematical Expressions: It is desirable that units of measurements and abbreviations should follow the System International (SI) except where the other unit system is more suitable. The numbers identifying the displayed mathematical expression should be placed in the parentheses and referred in the text as following examples (e.g., Eq. (1), Eqs. (1)-(2)). Mathematical expressions must be inserted as an object (set as Microsoft Equations 3.0) for Microsoft Word 2007 and later versions. Image-copied text or equations are not acceptable unless they are editable. The raised and lowered fonts cannot be used for superscription and subscription.

References : A list of references which reflect the current edition of APA format, to locate after conclusion of the paper.

Review
All the submitted papers will undergo a peer-review process, and those papers positively recommended by at least two expert reviewers will be finally accepted for publication in the Abacus Journals or after any required modifications are made and a payment of **₦15,000.00** as publication fee. The normal length of the paper should be between 8 to 12 pages. Any extra page will attract additional charges at the rate of **₦500.00** per page.

Copyright
Submission of an article to Abacus Journal implies that it presents the original and unpublished work, and not under consideration for publication elsewhere. On acceptance of the submitted manuscript, it is implied that the copyright thereof is transferred to the Abacus. The Transfer of Copyright Agreement may also be submitted.

Editorial Board

Associate Editors (2023-2025)

Editorial of Vol. 50, No. 1, April 2023 of ABACUS, the Journal of the Mathematical Association of Nigeria (MAN)

Exploring New Horizons in Mathematics

In the evolving landscape of scientific inquiry and technological advancement, mathematics remains the cornerstone of progress and innovation. As we present the first issue of the 50th volume of Abacus, we reflect on the journey of mathematics within Nigeria and the global context. This issue, marking a significant milestone in our publication history, brings together a collection of works that exemplify the vibrancy and depth of mathematical sciences today.

Welcome to the Volume 50, No. 1, April 2023 edition of **ABACUS**, the Journal of the Mathematical Association of Nigeria. As we embark on this milestone 50th volume, we reflect on the rich history and ongoing evolution of mathematics in Nigeria and beyond. This issue brings together a compelling collection of research articles, reviews, and insights that underscore the dynamic and transformative power of mathematical sciences.

Mathematics is often heralded as the language of the universe, a discipline that transcends borders and fuels innovation across a myriad of fields. In recent years, the importance of mathematical literacy and its applications has never been more apparent. From data science and artificial intelligence to epidemiology and environmental modeling, mathematics provides the critical frameworks and tools necessary to tackle some of the most pressing challenges of our time.

In this edition, we present a diverse range of topics that reflect the breadth and depth of contemporary mathematical research. Our contributors, comprising esteemed academics, researchers, and practitioners, offer new perspectives and findings that contribute to the ongoing dialogue in our field.

One highlight of this issue is a groundbreaking study on the application of machine learning algorithms in predicting climate change patterns. This research not only demonstrates the power of mathematical models in understanding complex environmental phenomena but also underscores the urgent need for interdisciplinary approaches to address global issues.

In the realm of applied mathematics, we feature studies on computational models that address real-world problems, such as climate modeling and optimization in industrial processes. These articles not only demonstrate the applicability of mathematical theories but also emphasize the interdisciplinary nature of modern scientific research. By bridging the gap between abstract concepts and tangible outcomes, these studies highlight the essential role of mathematics in solving complex global challenges.

Another featured article delves into the advancements in cryptographic techniques. As cybersecurity becomes increasingly critical in our digital age, the development of more robust encryption methods is paramount. This paper explores innovative approaches that enhance the security of data transmission and storage, with significant implications for both public and private sectors.

Additionally, we are excited to include a comprehensive review of recent developments in mathematical education in Nigeria. This piece provides a critical analysis of current pedagogical strategies, the integration of technology in the classroom, and the ongoing efforts to foster a deeper appreciation and understanding of mathematics among students at all levels. By highlighting successful initiatives and identifying areas for improvement, this review aims to contribute to the ongoing enhancement of mathematics education in our country.

Furthermore, we are pleased to include articles that discuss the role of mathematics in emerging fields such as data science and artificial intelligence. The exponential growth in data generation necessitates sophisticated mathematical tools for analysis and interpretation. The contributions in this volume provide cutting-edge research on algorithms and statistical models that are pivotal in advancing these technologies.

As we celebrate this milestone volume, we also look ahead to the future of mathematics in Nigeria. The Mathematical Association of Nigeria is committed to nurturing young talents and promoting a robust research culture. We are excited to announce initiatives aimed at supporting upcoming mathematicians through scholarships, workshops, and collaborative projects.

As we celebrate this landmark volume, we also look ahead to the future of ABACUS and the role it will continue to play in promoting mathematical scholarship. We remain committed to providing a platform for the dissemination of high-quality research, fostering collaboration among mathematicians, and inspiring the next generation of scholars.

We extend our deepest gratitude to our contributors, reviewers, and readers for their unwavering support and dedication to the advancement of mathematical sciences. Your collective efforts ensure that *Abacus* remains a beacon of knowledge and innovation.

In conclusion, we invite you to immerse yourself in the diverse and enriching content of this issue. Let us continue to push the boundaries of our understanding, explore new horizons, and harness the power of mathematics to shape a better future for all.

Sincerely,

Professor Muhammad Lawan Kaurangini
Editor-in-Chief
Abacus, Journal of the Mathematical Association of Nigeria

M.A.N. ABACUS

Abacus (Mathematics Science Series)
Vol. 50, No 1, April, 2023

AN OPTIMAL CONTROL STRATEGY FOR A SHIGELLA MODEL

Onoja Abu[1], Patrick Noah Okolo[2], Rose Abah[3]
[1]Department of Mathematics and Statistics, Federal Polytechnic, Idah, Nigeria
[2]Department of Mathematical Sciences, Kaduna University, Kaduna, Nigeria
[3]Department of Mathematics, University of Abuja, Abuja, Nigeria

Abstract

Shigellosis is an important public health problem as at least an estimated 140 million cases and over 500000 deaths are recorded annually across the globe. The attendant economic and financial burdens that accrue to the associated illnesses and deaths are much. The purpose of this study is to determine an optimal control strategy among some selected interventions. Therefore, an optimal control model for shigella epidemics incorporating two control variables representing antibiotic treatment; and personal hygiene and sanitation was formulated. The necessary optimality conditions were derived using Pontryagin's maximum principle. The corresponding optimality system was solved using Runge-Kutta method of order four. The results show that the model is able to determine an optimal strategy for management of shigellosis under some conditions.

Introduction

Shigellosis is a diarrheal disease caused by infection with the bacteria belonging to the genus Shigella with humans and primates as the only natural hosts [ECDC, (2015); WHO, (2005)]. Severity of illness caused by Shigella is a function of the type of species responsible for the infection. The four species of importance are: Shigella sonnei, S. flexneri, S. dysenteriae, and S. boydii [McCrickard et al. (2018)]. Shigellosis is transmitted by direct person-to-person contact, or by ingesting food or water contaminated with Shigella [ECDC (2015); WHO (2005)]. An infectious dose as low as 10-200 viable organisms suffices for person-to-person transmission [Ram et al. (2007); WHO (2005)].

Shigella infection on average manifests symptoms within four days after exposure. The common symptoms include bloody diarrhea, fever, abdominal pain, which usually disappear between five to seven days with or without antibiotic treatment [ECDC (2015); WHO (2005)].

Shigellosis is endemic in many developing countries and poses considerable attendant morbidity and mortality in these countries (WHO, 2005). Globally, it is estimated that Shigella breeds 140-165 million cases and 600,000-700,000 deaths annually [McCrickard et al. (2018); WHO (2005)]. The bulk of the global burden of Shigellosis is borne by developing countries in sub-Saharan Africa like Nigeria with dilapidated basic infrastructures [Ngoshe et al.(2018)]; and approximately 70% and 60% of cases and deaths occur among children aged less than five years [Ram et a. (2007); WHO (2005)].

Shigellosis flourishes in an environment that is characterized by poor personal hygiene, overcrowded living conditions, suboptimal sanitation and open sewage [Ngoshe et al. (2018)]. Preventive and control interventions for shigellosis are available. These include antibiotic treatment, proper personal hygiene and sanitation and proper sewage disposal [WHO (2005)].

Infectious diseases pose serious health and financial burdens to those countries that have been afflicted; these are being exacerbated with the emergence of new pathogens and re-emergence of old ones. In recent times, governments of the affected countries and supportive donor agencies have resorted to seeking cost-effective strategies for the management of infectious diseases. Mathematical models are veritable tools for studying the dynamics of infectious diseases. See, for example, Anderson and May (1991). Optimal control techniques have been used to determine best control strategies for infectious diseases such as malaria, Ebola, Influenza, tuberculosis, hepatitis B, tungiasis, to mention a few. See [Khamis et al. (2018); Otieno et al. (2016); Lashari et al. (2012); Nwanga et al. (2014); Ebenezer et al. (2016); Kahuru et al. (2017); Hattaf et al. (2009); Silver and Torries. (2014); Athithan and Gosh (2016); Tchuenche et al. (2011)]. Mathematical models for shigella transmission dynamics are scanty. However, the reader can see [Chaturvedi et al. (2014); Berhe et al. (2018) and Berhe et al. (2019)].

The goal of this paper is to develop an optimal control model for Shigella that can select an optimal and cost-effective strategy. The plan of this paper is as follows. Formulation of an optimal control model is done in section 2. Analysis of the optimal control model is done in section 3.Numerical simulations and results are presented in section 4. Cost-effectiveness analysis is carried out in section 5. Discussion of results is done in section 6. Finally the conclusive remarks are passed in section 7

Formulation of an Optimal Control Model

In this section, an optimal control model for shigella epidemics incorporating treatment, and personal hygiene and sanitation as controls is formulated. We considered human and shigella populations. The human population is partitioned into four compartments consisting of the susceptible humans (S) who are at risk of contracting infection, symptomatically infected humans (I) who are infectious and develop symptoms after infection, asymptomatically infected humans (C) who are infectious and may not develop symptoms and recovered humans (R). The shigella population is represented by B. Other parameters of the model are given in Table 1. Therefore, the optimal control model is given as follows.

The objective functional is given by

$$J(u) := \int_0^T [g(u)]dt = \int_0^T \left[A_1u_1I + A_2u_2S + \frac{1}{2}(a_1u_1^2 + a_2u_2^2) + dC_1I + C_2I\right] dt \qquad (1.1)$$

The constants A_1, A_2, A_3, a_1 and a_2 and C_1 and C_2 are cost weights associated with the infected humans, pathogens and the implementations of the controls.

The states for the model are given by the following system of differential equations.

$$\frac{dS}{dt} = \wedge + \alpha R - (I - u_2)(\lambda_h + \lambda_B)S - \mu S. \qquad (1.2)$$

$$\frac{dI}{dt} = \rho C + q(I - u_2)(\lambda_h + \lambda_B)S - (\mu + u_1 + \gamma + d)I. \qquad (1.3)$$

$$\frac{dC}{dt} = (1 - q)(I - u_2)(\lambda_h + \lambda_B)S - (\mu + \gamma_1 + \rho)C, \qquad (1.4)$$

$$\frac{dR}{dt} = (\gamma + u_1)I + \gamma_1 C - (\mu + \alpha)R \qquad (1.5)$$

$$\frac{dB}{dt} = \varepsilon_1 I + \varepsilon_2 C - (\sigma + ru_2)B, \qquad (1.6)$$

where $S(0) \geq 0, I(0) \geq 0, C(0) \geq 0, R \geq 0, B(0) \geq 0$, and the parameters $\lambda_h = \frac{\beta_{h1}I + \beta_{h2}C}{N}$ and $\lambda_B = \frac{\beta_B B}{K+B}$ represent rates of ingesting shigella from contaminated food or water and via human to human contact respectively (Berhe et al., 2018). The model is based on the following assumptions.

We assume that N is constant. The population is homogeneously mixed. The rate of recruitment of the susceptible humans is by birth or immigration (Berhe et al., 2018). The force of infection from human to human is assumed to be standard incidence and that from environment to human is logistic (Berhe et al., 2018).

Table 1: Table of Parameters

$parameter$	$Description$	$Units$
Λ	Recruitment rate of susceptible population	$Humans\ Time^{-1}$
K	shigella concentration that yields $25-50\%$ chance of catching dysentery diarrhea	$Cells\ Time^{-1}$
α	Relapse rate of the recovered humans to the susceptible class	$Time^{-1}$
λ_h	Effective transmission rate of diarrhea due to human to human interaction	$Time^{-1}$
λ_B	Effective transmission rate of dysentery diarrhea due to environment to human interaction	$Time^{-1}$
μ	Natural death rate of humans	$Time^{-1}$
ρ	Rate at which asymptomatic individuals develop symptoms	$Time^{-1}$
u_1	Treatment as a control variable	$Time^{-1}$
u_2	Personal hygiene and sanitation as a control variable	$Time^{-1}$
γ	Natural recovery rate of diarrhea	$Time^{-1}$
d	Disease induced death rate of dysentery diarrhea	$Time^{-1}$
q	Proportion of infections that are symptomatic	$Time^{-1}$
$(1-q)$	Proportion of infections that are asymptomatic	$Time^{-1}$
γ_1	Natural recovery rate of diarrhea	$Time^{-1}$
ε_1	Pathogen shedding rate	$Time^{-1}$
ε_2	Pathogen shedding rate	$Time^{-1}$
r	Efficacy of u_2	$Time^{-1}$
σ	Shigella Pathogen death rate	$Time^{-1}$

3. Analysis of the Optimal Control Problem

The Langragian is given by

$$L=\left[A_1u_1I+A_2u_2S+\frac{1}{2}(a_1u_1^2+a_2u_2^2)+dC_1I+C_2I\right]$$

The Hamiltonian is given by the sum of the Langragian and the inner product of the adjoint variables $\lambda_i$and the right-hand side of the state equations $(1.2)-(1.6)$; this is given by:

$H=L+\sum_{I=1}^{6}\lambda_I fi\,,fi$ As the right-hand side of equation (2)

$$H=\left[A_1u_1I+A_2u_2S+\frac{1}{2}(a_1u_1^2+a_2u_2^2)+dC_1I+C_2I\right]+\lambda_1[\Lambda+\alpha R-(1-u_2)(\lambda_h+\lambda_B)S$$
$$-\mu S]$$

$$+\lambda_2\,[\rho C+q(1-u_2)(\lambda_h+\lambda_B)S-(\mu+\gamma+u_1+d)I$$

$+\lambda_3[(1-q)(1-\mathrm{u}_2)(\lambda_h+\lambda_B)S-(\mu+\gamma_1+\rho)C]$

$+\lambda_4[(\gamma+u_1)I+\gamma_1\mathrm{C}-(\mu+\alpha)R]$

$+\lambda_5[\varepsilon_1 I+\varepsilon_2 C-(\sigma+ru_2)B$

The Optimality System

Suppose $U=(u_1,u_2)$ is a control vector, $x=(S,I,C,R,B)$ the state variables of the system $(1.2-1.6)$ and H the Hamiltonian, the optimality system is given by

$\frac{dx_i}{dt}=\frac{\partial H}{\partial\lambda_i}$, $-\frac{\mathrm{d}\lambda_i}{dt}=\frac{\partial H}{\partial x_i}$, $\mathrm{i}=1,\dots,5$ with transversality conditions $\lambda_i(tf)=0$; $\frac{\partial H}{\partial u_j}=0$, $\mathrm{j}=1,2$

Theorem 1. Let $U=(u_1,u_2)$ be a control vector, $x=(S,I,C,R,B)$ be the state variables of the system $(1.1-1.6)$ and H the Hamiltonian. There exists an optimal control vector $U^*(t)$ and the corresponding state vector $x^*(t)$ that minimize $J(U)$ over Ω. Furthermore, there exist adjoint functions λ_i satisfying the equations

$-\frac{d\lambda_i}{dt}=\frac{\partial H}{\partial x_i}$, $\mathrm{i}=1,\dots,5$ with transversality conditions $\lambda_i(tf)=0$.

In addition, the optimal controls are given by $u_j^*=\max\{0,\min(1,R_j)\}, j=1,2$.

Proof: We use the recipe by Berhe et al (2018). The existence of an optimal control vector follows from the convexity of the integrand J with respect to U, a priori boundedness of the state solutions and the Lipscitz property of the state solutions with respect to the state variables. See [Fleming and Rishel (1975)] (corollary 4.1). The adjoint equations and transversality conditions can be obtained by using the Pontryagin's Maximum Principle such that:

$$-\frac{d\lambda_1}{dt}=\frac{\partial H}{\partial S}=\lambda_1[\mu-(1-u_2)(\lambda_h+\lambda_B)]+\lambda_2[q(1-u_2)(\lambda_h+\lambda_B)]+$$
$$\lambda_3[(1-q)(1-u_2)(\lambda_h+\lambda_B)]+A_2u_2$$

$$-\frac{d\lambda_2}{dt}=\frac{\partial H}{\partial I}=-\lambda_1\left[(1-u_2)\frac{\beta_{h1}}{N}S\right]+\lambda_2\left[q(1-u_2)\frac{\beta_{h1}}{N}S-(\mu+\gamma+u_1+d)\right]+$$
$$\lambda_3(1-q)(1-u_2)\frac{\beta_{h1}}{N}S+\lambda_4(\gamma+u_1)+\lambda_5\varepsilon_1+A_1u_1+C_1+C_2$$

$$-\frac{d\lambda_3}{dt}=\frac{\partial H}{\partial C}=-\lambda_1[(1-u_3)\frac{\beta_{h2}}{N}S]+\lambda_2 q(1-u_2)\frac{\beta_{h2}}{N}S+\lambda_2\rho+\lambda_3[(1-q)(1-u_2)\frac{\beta_{h1}}{N}S$$
$$-(\mu+\gamma_1+\rho)]+\lambda_4\gamma_1+\lambda_5\varepsilon_2$$

$$-\frac{d\lambda_4}{dt}=\frac{\partial H}{\partial R}=\lambda_1\alpha-\lambda_4(\mu+\alpha)$$

$$-\frac{d\lambda_5}{dt}=\frac{\partial H}{\partial B}=-\lambda_1(1-u_2)\left[\frac{(K+B)\beta_B-\beta_B B}{(K+B)^2}\right]+\lambda_2(1-u_2)\left[\frac{(K+B)\beta_B-\beta_B B}{(K+B)^2}\right]$$
$$+\lambda_3(1-q)\left[\frac{(K+B)\beta_B-\beta_B B}{(K+B)^2}\right]-\lambda_5(\sigma+ru_2)$$

with transversality conditions $\lambda_1(\mathrm{T})=\lambda_2(T)=\lambda_3(T)=\lambda_4(T)=\lambda_5(T)=0$

The controls u_j can be solved for by using the optimality condition

$$\frac{\partial H}{\partial u_1}=0;\ \frac{\partial H}{\partial u_2}=0;\ \text{Therefore,}$$

$$u_1^*=\max\{0,\min(1,R_1)\},\ R_1=\frac{(\lambda_2-\lambda_4)I-A_1 I}{a_1}$$

$$u_2^*=\max\{0,\min(1,R_2)\},\ R_2=\frac{\lambda_2[(\lambda_h+\lambda_B)-\lambda_1[(\lambda_h+\lambda_B)S]]+\lambda_5 rB-A_2 S}{a_2}$$

Numerical Simulation and Results

Table 2: Table of parameter values

parameter	*values*	*soutce*
Λ	2	
K	200	Berhe et al.(2019)
α	0.0030	Berhe et al.(2019)
λ_h	0.0998	Berhe et al.(2019)
λ_B	0.0064	Berhe et al.(2019)
μ	0.0000547	Estimated
ρ	0.01	Assumed
u_1	0-1	Variable
u_2	0-1	Variable
γ	0.0158	Variable
d	0.0794	Berhe et al.(2019)
q	0.5	Assumed
$(1-q)$	0.5	Assumed
γ_1	0.0158	Berhe et al.(2019)
ε_1	10	Assumed
ε_2	10	Assumed
r	0.25	
σ	0.01	
A_1	500	Estimated
A_2	300	Estimated
A_3	10	Estimated
a_1	100	Estimated
a_2	750	Estimated

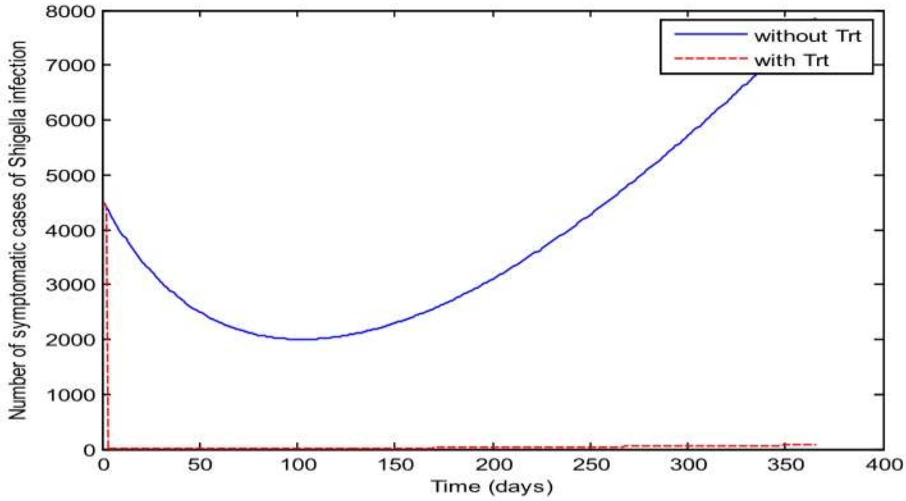


Figure 1: The plot represents populations of symptomatic infected humans with Treatment (u_1) and without control.

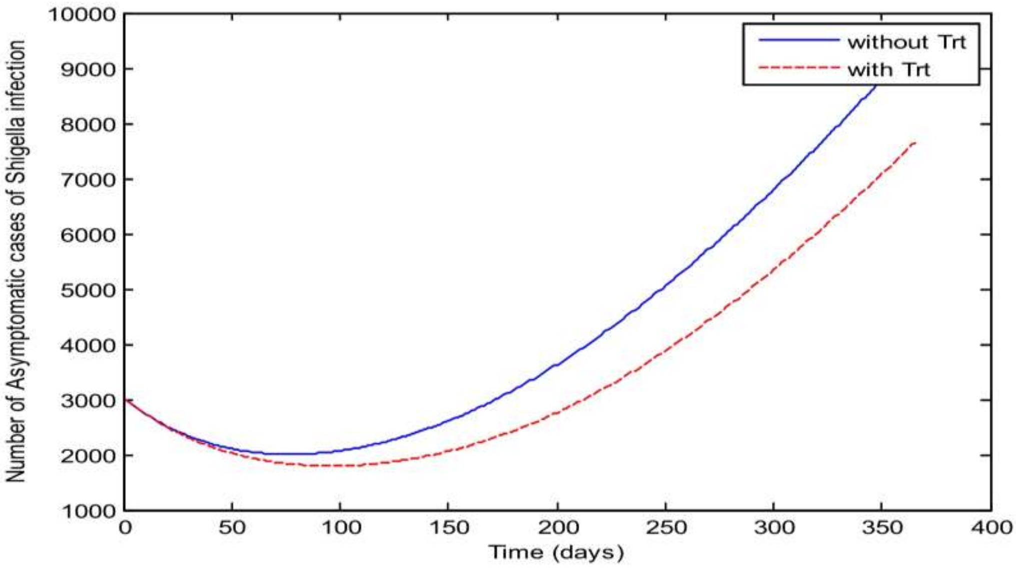


Figure 2: The plot represents populations of asymptomatic infected humans with Treatment (u_1) and without control.

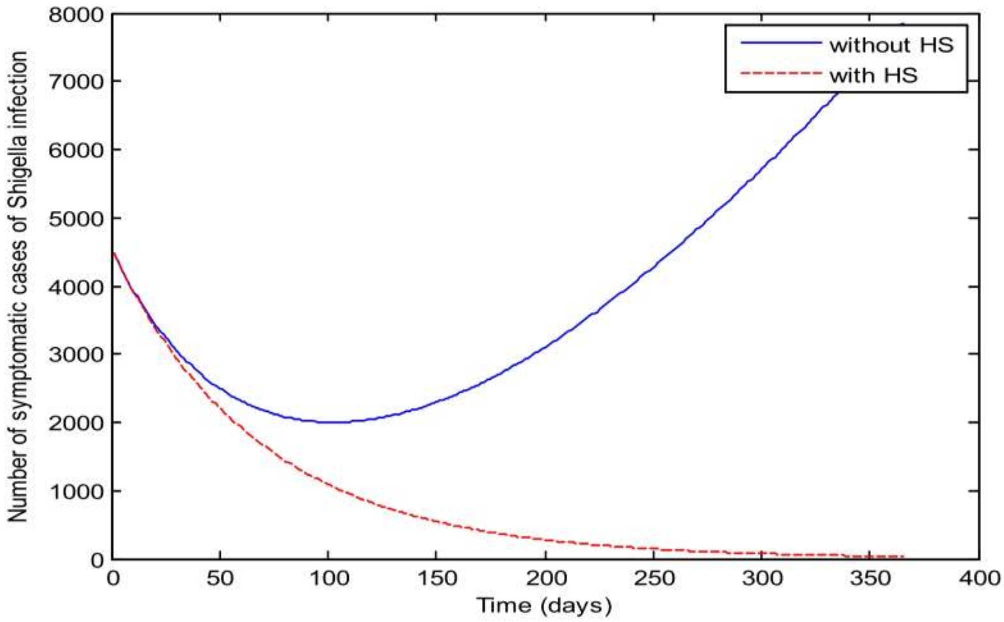


Figure 3: The plot represents populations of symptomatic infected humans with hygiene and sanitation (u_2) and without control.

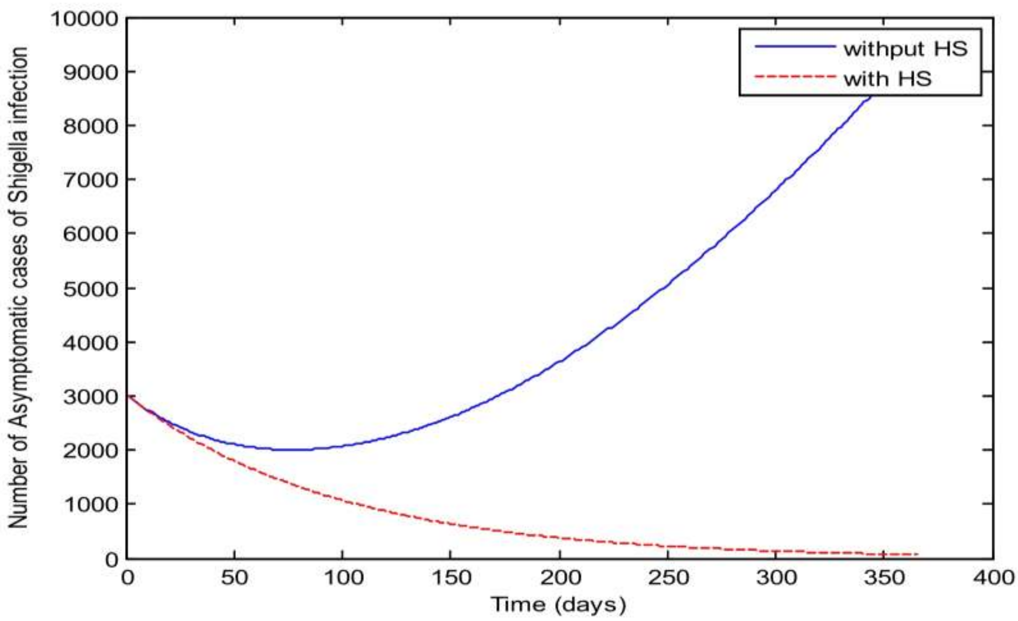


Figure 4: The plot represents populations of asymptomatic infected humans with hygiene and sanitation (u_2) and without control.

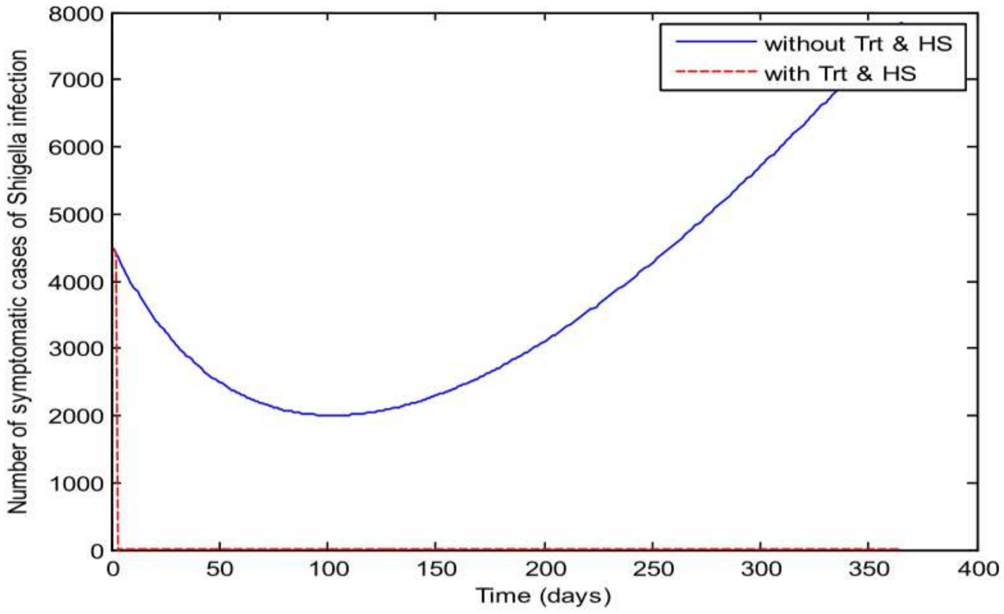


Figure 5: The plot represents populations of symptomatic infected humans with treatment, and hygiene and sanitation (u_1, u_2) and without control.

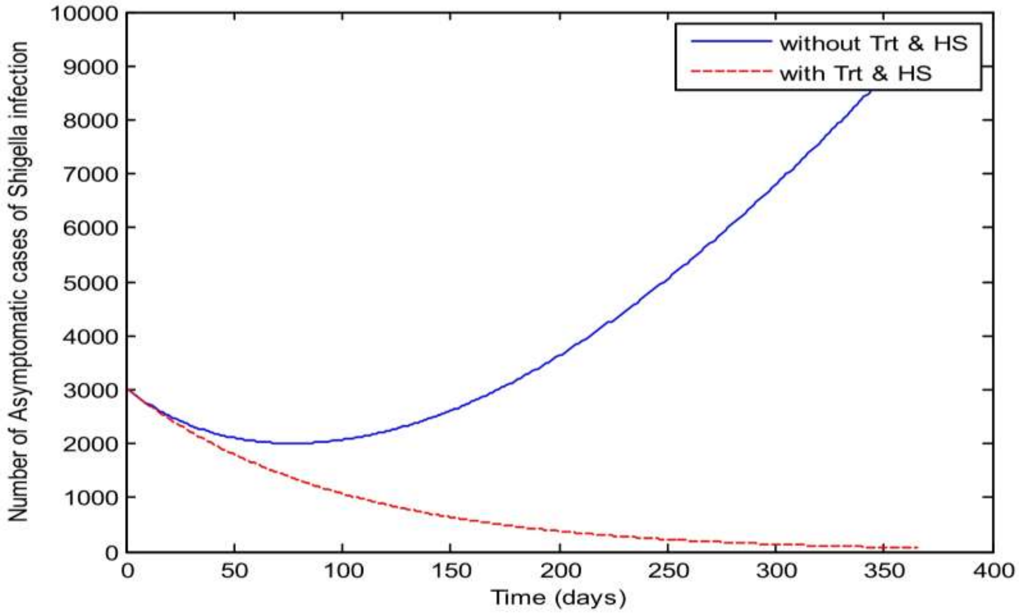
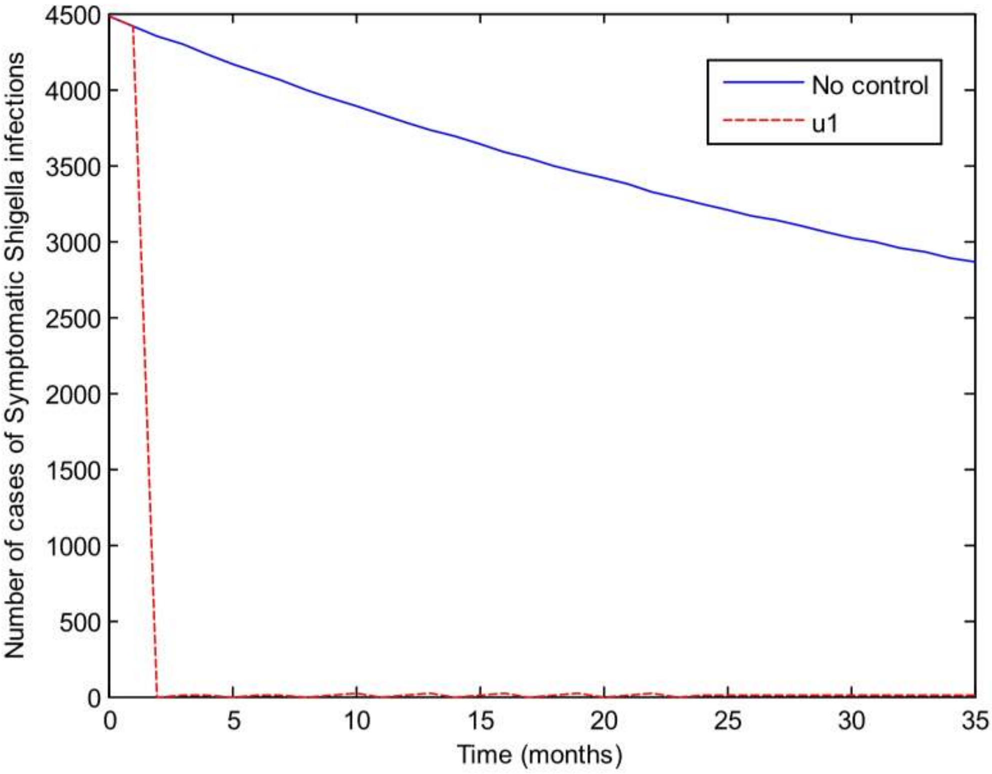
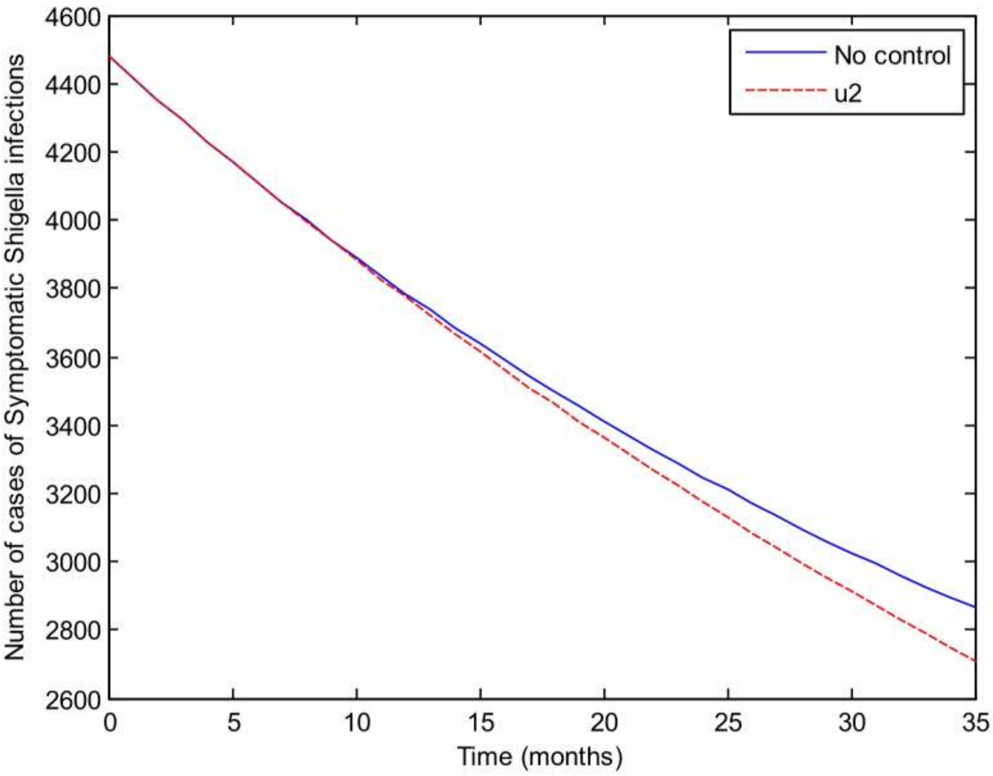
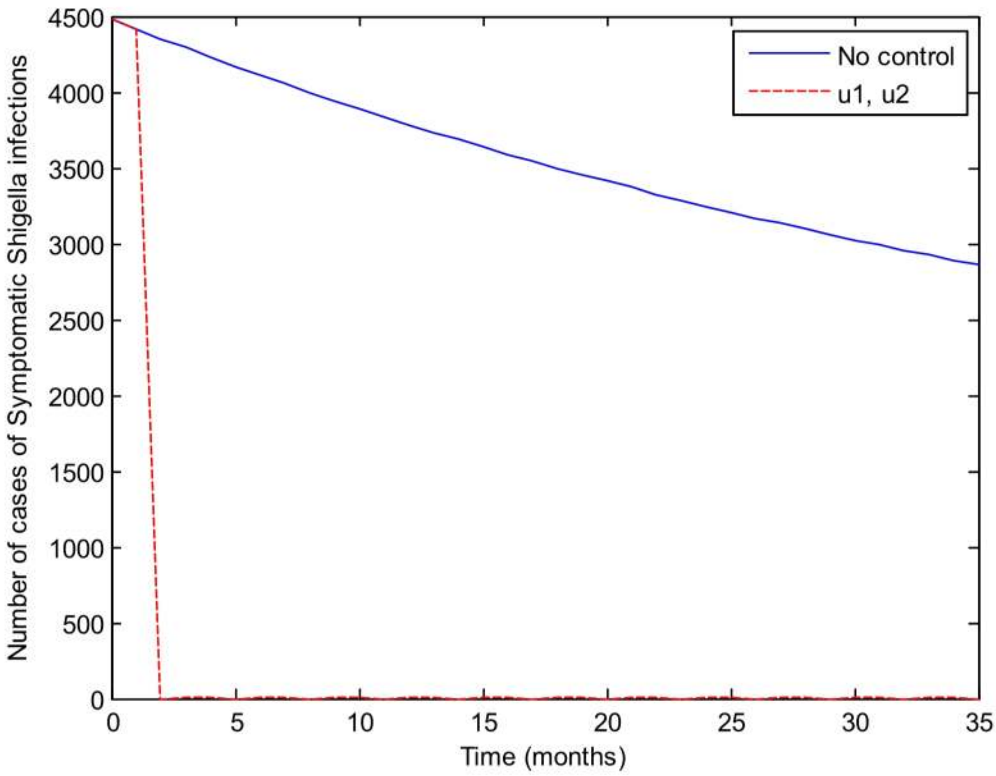


Figure 6: The plot represents populations of asymptomatic infected humans with treatment, and hygiene and sanitation (u_1, u_2) and without control.


Number of cases of Symptomatic Shigella infections
Time (months)
No control
u1
4500
4000
3500
3000
2500
2000
1500
1000
500
0
5
10
15
20
25
30
35


Number of cases of Symptomatic Shigella infections
Time (months)
No control
u2
4600
4400
4200
4000
3800
3600
3400
3200
3000
2800
2600
0
5
10
15
20
25
30
35



Cost-Effectiveness Analysis

In this section, the method of incremental cost-effective ratio (ICER) is used to compare cost-effectiveness of two strategies. The cost objective functional is used to evaluate the total costs associated with all possible strategies over the period. The numbers of infections averted and the total costs of the corresponding strategies are shown in Table 3.

Table 3

Strategy	Description	No. of infections averted	Total cost of control
A	Treatment	119565	161134.60
B	Hygiene and sanitation	1800	1421137.90
C	Treatment, and Hygiene and sanitation	119641	171519.28

We compare the cost-effectiveness of strategies pair wise as follows.

$ICER(B) = \frac{1421137.90}{1800} = 789.52,\ ICER(A) = \frac{1421137.90-161134.60}{1800-119565} = -10.70$

This shows that strategy A is cheaper than strategy B.

$$ICER(A) = \frac{161134.60}{119565} = 1.35, \qquad ICER(C) = \frac{161134.60 - 171519.28}{119565 - 119641} = 136.64$$

This shows that A is better than strategy C.

Discussion

This section discusses analytical and numerical results of our optimal control model. The optimal control model incorporates two time-dependent variables consisting of treatment (Trt), and personal hygiene and sanitation. The main analytical result on existence and characterization of the corresponding optimality system of the control model can be found in Theorem 1. The optimality system was numerically explored for three different possible intervention strategies. Table 3 shows the costs implications of the three different strategies while Figures 1 through 6 show the best outcomes of our numerical experiments. Table 3 shows that treatment and hygiene/sanitation as a control incurs the highest cost. The total cost of this intervention based on our model is 171519.28. In another development, treatment as a strategy incurs the lowest cost of 161134.60. The effects of this strategy on infected human population are depicted in Figures 1 through 6.

Conclusion

In this paper, we formulated an optimal control model for shigella epidemics, bearing in mind, the role of asymptomatic infections on the transmission dynamics of shigella. The main analytical result on existence and optimality system can be found in Theorem 1. Important numerical results can be found in Figures 1 through 6. The costs of the three different possible strategies can be seen in Table 3. The results show that optimal use of treatment as a control strategy eliminates shigellosis fastest and gives the lowest cost. Optimal use of treatment and hygiene/sanitation is the most cost-effective and therefore, recommended in endemic and epidemic settings.

References

Athithan, S. and Ghosh, M. (2016) Optimal control of tuberculosis with case detection and treatment, WJMS, 11(2): 111-122.

Anderson, R.M. and May, R.M. (1991). Infectious diseases of Humans: Dynamics and Control, Oxford University Press.

Berhe, H. W., Makinde, O. D. and Theuri, D. M. (2018). Optimal Control and Cost-Effectiveness Analysis for Dysentery Epidemic Model, Appl. Math. Inf. Sci. 12, No. 6, 1183-1195 (2018)

Berhe, H. W., Makinde, O. D. and Theuri, D. M. (2019) Parameter Estimation and Sensitivity Analysis of Dysentery Diarrhea Epidemic Model, Journal of Applied Mathematics, 1-13

Chaturvedi, O., Masupe, T. and Masupe, S. (2014). A Continuous Mathematical Model for Shigella Outbreaks, American Journal of Biomedical Engineering, 4(1): 10-16

Ebenezer, B., Badu, K. and Kwesi, A, S, (2016). Optimal Control Application to an Ebola Model, Health Science Journal, 10(2:7): 1-7.

European Centre for Disease Prevention and Control (ECDC) (2015). Shigellosis among refugees in the EU – 26 November 2015, Stockholm: ECDC.

Fleming, W.H. and Rishel, R.W.,(1975). Deterministic and Stochastic Optimal Control, Wiley.

Hattaf, K. Rachik, M.., Saadi, S., Tabit, Y. and Yousfi, N. (2009). Optimal Control of Tuberculosis with Exogenous Reinfection, Applied Mathematical Sciences, 3(5):231-240.

Kahuru, J. Luboobi, L. S. and Nkansah-Gyekye1, Y. (2017). Optimal Control Techniques on a Mathematical Model for the Dynamics of Tungiasis in a Community, International Journal of Mathematics and Mathematical Sciences, Vol. 2017, 1-20.

Khamis, D., Mouden, C. E., Kura, K. and Bonsall, M. B. (2018). Optimal control of malaria: combining vector interventions and drug therapies, Malar J, 17(174):1-18.

Lashari, A. A., Aly, S. Hattaf, K., Zaman, G.. Jung, H and. Li, X. Z (2012). Presentation of Malaria Epidemics Using Multiple Optimal Controls, Journal of Applied Mathematics,

McCrickard , L. S. Crim, S. M. Kim, S. and Bowen, A., (2018). Disparities in severe shigellosis among adults — Foodborne diseases active surveillance network, 2002–2014, BMC Public Health, 18:221

Ngoshe, I. Y.. Denue, B. A., Bello, H. S., Akawu, C. B. Gashau, W. (2018). Prevalence and antimicrobial susceptibility of shigella spp in northeastern Nigeria, 2018 Sub-Saharan African Journal of Medicine | Published by Wolters Kluwer – Medknow

Mwanga, G. G. , Haario, H. Nannyonga, B. K. (2014). Optimal Control of Malaria Model with Drug Resistance in Presence of Parameter Uncertainty, Applied Mathematical Sciences, 8(55): 2701 – 2730.

Otieno, G., Koske, J. K. and.Mutiso, J. M. (2016). Transmission Dynamics and Optimal Control of Malaria in Kenya, Hindawi Discrete Dynamics in Nature and Society, Volume 2016, 1-27.

Ram, P. K, Crump, J. A., Gupta, S. K., Miller, M. A. and Mintz, E. D. (2007). Analysis of data gaps pertaining to Shigella infections in low and medium human development index countries, 1984–2005

Silva, C. J. and Torres, D. F. M. (2014). Optimal Control of Tuberculosis: A Review, Mathematics of the Planet Earth, 1-22,

Tchuenche, J. M. Khamis, S. A., Agusto, F. B. and Mpeshe, S. C. (2011). Optimal Control and Sensitivity Analysis of an Influenza Model with Treatment and Vaccination, Acta Biotheor, 59:1–28.

WHO (2005). Guidelines for the control of shigellosis, including epidemics due to Shigella dysenteriae 1. World Health Organization 2005, Geneva

CONTINUOUS FORMULATION OF HYBRID BLOCK MILNE TECHNIQUE FOR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

K. J. Audu[1*], Y. A. Yahaya[2], J. Garba[3], A. T. Cole[4] and F. U. Tafida[5]
[1,2,3,4]Department of Mathematics, Federal University of Technology, P.M.B. 65 Minna, Nigeria
[5]Department of Mathematics, Ibrahim Badamasi Babangida University, P.M.B. 11, Lapai, Nigeria
email: k.james@futminna.edu.ng

Abstract

In most Scientific and engineering problems, ordinary differential equations cannot be solved by analytic methods. Consequently, numerical approaches are frequently required. A block hybrid Milne technique was formulated in this paper in order to develop a suitable algorithm for the numerical solution of ordinary differential equations. Utilizing power series as the basis function, the proposed method is developed. The developed algorithm is used to solve systems of linear and nonlinear differential equations, and it has proven to be an efficient numerical method for avoiding time-consuming computation and simplifying differential equations. The fundamental numerical properties are examined, and the results demonstrate that it is zero-stable and consistent, which ensures convergence. In addition, by comparing the approximate solutions to the exact solutions, we demonstrate that the approximate solutions converge to the exact solutions. The results demonstrate that the developed algorithm for solving systems of ordinary differential equations is straightforward, efficient, and faster than the analytical method.

Keywords: Ordinary differential equations, numerical solution of ODEs, Hybrid Milne method, approximate solutions, algorithm and power series

Introduction

An equation in mathematics that describes the relationship between a function and its derivative is an example of a differential equation. In practical contexts, functions are typically used to represent rates of change. Engineers, physicists, economists, biologists, and others rely heavily on differential equations. Initial value first order ordinary differential equations appear in the process of modeling real-world situations in physical and applied sciences, particularly in algebraic expressions concerning problems related to flow of viscous thin films, disease models, chemical kinetics, quantum mechanics and electromagnetic waves (Aslam *et al.*, 2021; Mazarina and Syahirbanun (2022); Amat *et al.*, 2019; Kwanamu *et al.*, 2021). Understanding the behaviors and propertiesof the investigated physical phenomena requires the resolution of this type of problem (Kashkaria and Syam (2019)). In the majority of instances, available analytical approaches fail to provide an accurate solution to a general first-order initial value problem. To solve such problems that come up in various area of engineering and science, it is important to use numerical approaches that are close to the equations' solutions (Chapra and Canale, 2015). As such, scientific and technological problems involving differential equations are typically solved using numerical methods rather than analytic ones.

In this research, we intend to develop and study a four-step first derivative hybrid block Milne approach for systems of ordinary differential equations taken into account as:

$$\begin{cases} z_1' = f(t, z_1, z_2, \ldots, z_n) \\ z_2' = f(t, z_1, z_2, \ldots, z_n) \\ \vdots \\ z_n' = f(t, z_1, z_2, \ldots, z_n) \end{cases} \quad \text{with initial conditions} \quad \begin{cases} z_1(0) = c_1 \\ z_2(0) = c_2 \\ \vdots \\ z_n(0) = c_n \end{cases} \tag{1}$$

For arbitrary $z_0 \leq z \leq z_N$. In this case, the function $f(t, z)$ is assumed to be continuous throughout the integration interval, and a unique solution exists. Numerous research has been carried out to provide numerical solutions to problems modeled as first order ordinary differential equations. These include works of authors such as Ndipmong and Udechukwu (2022), Garba and Mohammed (2020), Gomathi and Rabiyabe (2022), Badmus *et al.* (2015), Ehiemua and Agbeboh (2019), Eziokwu and Okereke (2020). Iyorter, B. V., Luga, T. & Isah, S. S. (2019). Techniques of solution employed by the above researchers include, Euler methods, the Adams Bashforth and Adams Moulton methods, linear multistep methods, Runge-Kutta methods and Milne methods among others.

Few mathematicians have come up with some block Milne techniques regarding solutions to various differential problems. The convergence of some selected properties with respect to block predictor-corrector methods and its applications on differential problems were investigated (Oghonyon *et al*., 2016a). Again, Oghonyon *et al.,* (2016b) focused on block predictor-corrector method and derived a Milne's scheme. They implemented the scheme on ordinary differential problems and obtained a favourable outcome. Recently, Oghonyon *et al.* (2018a) formulated a suitable exponential fitted block Milne's scheme for ordinary differential equations emerging from oscillating vibrations problems. However, these approaches are limited by their low accuracy rate and low number of steps. The present research was motivated by the need to overcome the shortcomings of existing approaches by expanding the number of steps at both grid and off-step locations. The Milne technique employs the predictor-corrector algorithm and is dahlquist stable and accurate to the second order. For their starting values, the predictor-corrector of the Milne scheme requires single-step methods. In this study, the corrector component is reformulated into a continuous form and implemented as a block method in order to make it self-starting to solve systems of ordinary differential equations. To improve the degree of accuracy of the Milne method, appropriate off-grid points are selected with care.

This paper is structured as follows. In Section two, we describe the construction of the new numerical technique for (1). In section three, we established the order, zero stability, consistency, and convergence of the technique. In Section four, we used the method to solve systems of differential equations of the first order and compared the results of the different problems. Numerical tests with sample problems and their results were resented in section 4 and we concluded the study in section 5.

Construction of the Block Hybrid Milne Technique

To derive the new numerical technique, we apply the notion of a linear multistep collocation procedure using the general format

$$z(t)=\sum_{n=0}^{s+1} A(t)z(t_{i+n}) = h\sum_{n=0}^{v-1} B(t)f(\overline{t}_n, \overline{z}(\overline{t}_n)) \tag{2}$$

where

$$A_i(t)=\sum_{n=0}^{u+v-1} A_{i,n+i}t^n \quad and \quad hB_i(t)=\sum_{n=0}^{w+v-1} B_{i,n+i}t^n \tag{3}$$

Here, we use the basis function of power series to derive a numerical estimate for the ordinary differential equation of the format described in (1).

$$\sum_{n=0}^{w+v-1} d_n t^n \tag{4}$$

where w and v represents the interpolation and collocation points, $t \in [t_0, z_N]$, and d_n's are unknowns. Equation (1) is differentiated to get

$$\sum_{n=0}^{w+v-1} n d_n t^{n-1} \tag{5}$$

Hence, the continuous format of the proposed block technique from (3) with five off grid points at collocation is represented as

$$\begin{aligned} z(t) = {} & A_2(t)z_{i+2} + B_2(t)hf_{i+2} + B_{\frac{9}{4}}(t)hf_{i+\frac{9}{4}} + B_{\frac{5}{2}}(t)hf_{i+\frac{5}{2}} + B_3(t)hf_{i+3} + B_{\frac{13}{4}}(t)hf_{i+\frac{13}{4}} \\ & + B_{\frac{7}{2}}(t)hf_{i+\frac{7}{2}} + B_4(t)hf_{i+4} \end{aligned} \tag{6}$$

It generated some non-linear system of equations in the format $Mt = B$ in (7)

$$\begin{pmatrix} 1 & (t_{i+2}) & (t_{i+2})^2 & (t_{i+2})^3 & (t_{i+2})^4 & (t_{i+2})^5 & (t_{i+2})^6 & (t_{i+2})^7 \\ 0 & 1 & 2(t_{i+2}) & 3(t_{i+2})^2 & 4(t_{i+2})^3 & 5(t_{i+2})^4 & 6(t_{i+2})^5 & 7(t_{i+\frac{9}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{9}{4}}) & 3(t_{i+\frac{9}{4}})^2 & 4(t_{i+\frac{9}{4}})^3 & 5(t_{i+\frac{9}{4}})^4 & 6(t_{i+\frac{9}{4}})^5 & 7(t_{i+\frac{9}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{5}{2}}) & 3(t_{i+\frac{5}{2}})^2 & 4(t_{i+\frac{5}{2}})^3 & 5(t_{i+\frac{5}{2}})^4 & 6(t_{i+\frac{5}{2}})^5 & 7(t_{i+\frac{5}{2}})^6 \\ 0 & 1 & 2(t_{i+3}) & 3(t_{i+3})^2 & 4(t_{i+3})^3 & 5(t_{i+3})^4 & 6(t_{i+3})^5 & 7(t_{i+3})^6 \\ 0 & 1 & 2(t_{i+\frac{13}{4}}) & 3(t_{i+\frac{13}{4}})^2 & 4(t_{i+\frac{13}{4}})^3 & 5(t_{i+\frac{13}{4}})^4 & 6(t_{i+\frac{13}{4}})^5 & 7(t_{i+\frac{13}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{7}{2}}) & 3(t_{i+\frac{7}{2}})^2 & 4(t_{i+\frac{7}{2}})^3 & 5(t_{i+\frac{7}{2}})^4 & 6(t_{i+\frac{7}{2}})^5 & 7(t_{i+\frac{7}{2}})^6 \\ 0 & 1 & 2(t_{i+4}) & 3(t_{i+4})^2 & 4(t_{i+4})^3 & 5(t_{i+4})^4 & 6(t_{i+4})^5 & 7(t_{i+4})^6 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{pmatrix} = \begin{pmatrix} z_{i+2} \\ f_{i+2} \\ f_{i+\frac{9}{4}} \\ f_{i+\frac{5}{2}} \\ f_{i+3} \\ f_{i+\frac{13}{4}} \\ f_{i+\frac{7}{2}} \\ f_{i+4} \end{pmatrix} \tag{7}$$

Employing Maple 2015 software to compute (7), and evaluation of the desired points results into the following proposed schemes;

$$z_{i+1} = -\frac{45373}{1260}hf_{i+2} + \frac{48518}{315}hf_{i+3} - \frac{3292832}{945}hf_{i+4} - \frac{134941}{945}hf_{i+\frac{5}{2}} + \frac{3035}{63}hf_{i+\frac{7}{2}} + \frac{269296}{2205}hf_{i+\frac{9}{4}} - \frac{27248}{189}hf_{i+\frac{13}{4}} + z_{i+2} \quad (8)$$

$$z_i = z_{i+2} - \frac{72353}{105}hf_{i+2} + \frac{1994752}{735}hf_{i+\frac{9}{4}} - \frac{3292832}{945}hf_{i+\frac{5}{2}} + \frac{443852}{105}hf_{i+3} - \frac{778240}{189}hf_{i+\frac{13}{4}} + \frac{149216}{105}hf_{i+\frac{7}{2}} - \frac{482819}{6615}hf_{i+4} \quad (9)$$

$$z_{i+\frac{9}{4}} = \frac{2251}{26880}hf_{i+2} + \frac{1661}{26880}hf_{i+3} - \frac{293}{423360}hf_{i+4} - \frac{23021}{241920}hf_{i+\frac{5}{2}} + \frac{85}{5376}hf_{i+\frac{7}{2}} + \frac{5549}{23520}hf_{i+\frac{9}{4}} - \frac{311}{6048}hf_{i+\frac{13}{4}} + z_{i+2} \quad (10)$$

$$z_{i+\frac{5}{2}} = \frac{787}{10080}hf_{i+2} + \frac{73}{2520}hf_{i+3} - \frac{83}{211680}hf_{i+4} + \frac{863}{15120}hf_{i+\frac{5}{2}} + \frac{43}{5040}hf_{i+\frac{7}{2}} + \frac{781}{2205}hf_{i+\frac{9}{4}} - \frac{5}{189}hf_{i+\frac{13}{4}} + z_{i+2} \quad (11)$$

$$z_{i+3} = \frac{37}{420}hf_{i+2} + \frac{44}{105}hf_{i+3} - \frac{47}{26460}hf_{i+4} + \frac{331}{945}hf_{i+\frac{5}{2}} + \frac{1}{21}hf_{i+\frac{7}{2}} + \frac{208}{735}hf_{i+\frac{9}{4}} - \frac{176}{945}hf_{i+\frac{13}{4}} + z_{i+2} \quad (12)$$

$$z_{i+\frac{13}{4}} = \frac{1405}{16128}hf_{i+2} + \frac{8875}{16128}hf_{i+3} - \frac{125}{84672}hf_{i+4} + \frac{16375}{48384}hf_{i+\frac{5}{2}} + \frac{575}{16128}hf_{i+\frac{7}{2}} + \frac{4075}{14112}hf_{i+\frac{9}{4}} - \frac{295}{6048}hf_{i+\frac{13}{4}} + z_{i+2} \quad (13)$$

$$z_{i+\frac{7}{2}} = \frac{99}{1120}hf_{i+2} + \frac{141}{280}hf_{i+3} - \frac{17}{7840}hf_{i+4} + \frac{197}{560}hf_{i+\frac{5}{2}} + \frac{15}{112}hf_{i+\frac{7}{2}} + \frac{69}{245}hf_{i+\frac{9}{4}} + \frac{1}{7}hf_{i+\frac{13}{4}} + z_{i+2} \quad (14)$$

$$z_{i+4} = \frac{17}{315}hf_{i+2} + \frac{404}{315}hf_{i+3} - \frac{869}{6615}hf_{i+4} + \frac{32}{945}hf_{i+\frac{5}{2}} + \frac{352}{315}hf_{i+\frac{7}{2}} + \frac{1024}{2205}hf_{i+\frac{9}{4}} - \frac{1024}{945}hf_{i+\frac{13}{4}} + z_{i+2} \quad (15)$$

Analysis of the Proposed Technique

This section is concerned on analyses with respect to zero stability and consistency of the novel technique.

Consistency

The proposed technique described in section 2 is frequently written as;

$$\sum_{i=0}^{4} \mathrm{A}_i z_{n+i} - \sum_{i=0}^{4} h\mathrm{B} f_{n+i} = 0 \tag{16}$$

Following Oghonyon *et al.* (2018b) and Mohammed *et al.* (2021), the local truncation error is a linear difference operator as; $L[z(t);h] = z_i - h\Big(\mathrm{B}_2(t)z'_n + \mathrm{B}_3(t)z'_n + \mathrm{B}_{\frac{5}{2}}(t)z'_n + \mathrm{B}_{\frac{7}{2}}(t)z'_n + \mathrm{B}_{\frac{9}{4}}(t)z'_n + \mathrm{B}_{\frac{13}{4}}(t)z'_n + \mathrm{B}_4(t)z'_n\Big)$ (17)

Assuming that $z(t)$ is sufficiently differentiable, then the Taylor's expansion of (17) about the point t, can be represented as;

$$L[z(t);h] = E_0 z(t) + E_1 h z'(t) + E_2 h^2 z''(t) + \cdots + E_p h^p z^p(t) + E_{p+1} h^{p+1} z^{p+1}(t) \quad (18)$$

The discrete scheme in (9) is said to be consistent if $p \geq 1$ for $E_0 = E_1 = E_2 = \cdots = E_p = 0, E_{p+1} \neq 0$, where E_{p+1} denotes the error constant, and p denotes the order of the hybrid technique (Tiamiyu *et al.,* 2021). The summary of the order and error constant of the block schemes is given in Table 1.

Table 1 –Error Constants and Order of the Proposed Technique

Equation	Order	Error constant
(8)	7	$\frac{1643}{15680}$
(9)	7	$\frac{70099}{27095040}$
(10)	7	$\frac{1051}{2055208960}$
(11)	7	$\frac{9925}{11098128384}$
(12)	7	$\frac{1}{1003520}$
(13)	7	$\frac{139}{433520640}$
(14)	7	$\frac{17}{16056320}$
(15)	7	$-\frac{1}{211680}$

Zero Stability

To determine the zero stability of the new derived schemes, the first characteristic polynomial $R(\lambda)$ of (8) to (15) denoted as $\det(\lambda \times A(1) - A(0))$ is normalized as follows;

$R(\lambda) = \det(\lambda \times A(1) - A(0))$ such that we obtain

$$R(\lambda) = \lambda \times \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (\lambda^2 - \lambda)\lambda^6$$

for $|\lambda| \leq 1$ and the roots $|\lambda| = 1$, the multiplicity must not exceed one. Hence, we arrive at the deduction $R(\lambda) = \det(\lambda \times A(1) - A(0)) = (\lambda^2 - \lambda)\lambda^6 = 0$ and $\lambda = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$. Therefore, the developed hybrid block Milne technique is said to be zero stable.

Convergence

According to Ma'ali *et al.* (2020), Dahlquist's fundamental theorem asserts that "the necessary and sufficient requirements for a linear multi-step procedure to be convergent are consistency and zero-stability. By Kashkaria and Syam (2019) and Oghonyon *et al.* (2018b), since the hybrid block approach provided is consistent and zero stable, the convergence requirement is met.

Numerical Tests

Problem 1: We consider a set of linear differential equations in the form;

$$\begin{aligned} z_1' &= -21z_1 + 19z_2 - 20z_3, \qquad & z_1(0) &= 1 \\ z_2' &= 19z_1 - 21z_2 + 20z_3, \qquad & z_2(0) &= 0 \\ z_3' &= 40z_1 - 40z_2 - 40z_3, \qquad & z_3(0) &= -1 \\ 0 \leq t &\leq 3, \qquad h = 0.2 \end{aligned}$$

The exact solution is provided as

$$z_1(t)=\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-40t}\sin(40t)+\frac{1}{2}e^{-40t}\cos(40t)$$
$$z_2(t)=\frac{1}{2}e^{-2}-\frac{1}{2}e^{-40t}\sin(40t)-\frac{1}{2}e^{-40t}\cos(40t)$$
$$z_3(t)=e^{-40t}\sin(40t)-e^{-40t}\cos(40t)$$

Problem 2: Considering the systems of initial value problem of first order differential equation of the form;

$$z_1'=-z_1+95z_2, \qquad z_1(0)=1$$
$$z_2'=-z_1-97z_2, \qquad z_2(0)=1$$
$$h=0.0625$$

The real solution is provided as;

$$z_1(t)=\frac{95}{47}e^{-2t}-\frac{48}{47}e^{-96t}$$
$$z_2(t)=\frac{48}{47}e^{-96t}-\frac{1}{47}e^{-2t}$$

Problem 3: We consider the systems of initial value problem of first order differential equation of the form;

$$z_1'=-\left(2+10^4\right)z_1+10^4 z_2, \qquad z_1(0)=1$$
$$z_2'=z_1-z_2-z_2^2, \qquad z_2(0)=1$$

With $h=0.1$ and the exact solution given as

$$z_1(t)=e^{-2t}$$
$$z_2(t)=e^{-t}$$

Problem 4. Solving the non-linear system of initial value problem of first order differential equation of the form;

$$z_1'=-1002z_1+100z_2{}^2, \qquad z_1(0)=1$$
$$z_2'=z_1-z_2(1+z_2), \qquad z_2(0)=1$$

$0\le t\le 1$ and the exact solution can be obtained from the following relations

$z_1(t)=e^{-2t},\ z_2(t)=e^{-t}$

Test Results

This section presents the test results for problems 1 to 4 considered in previous section. Comparison of the computations are displayed in some Figures and Tables. The exact solutions are represented by $z(t)$ and the new hybrid Block Milne solutions are denoted as $z_s(t), s=1,2$.

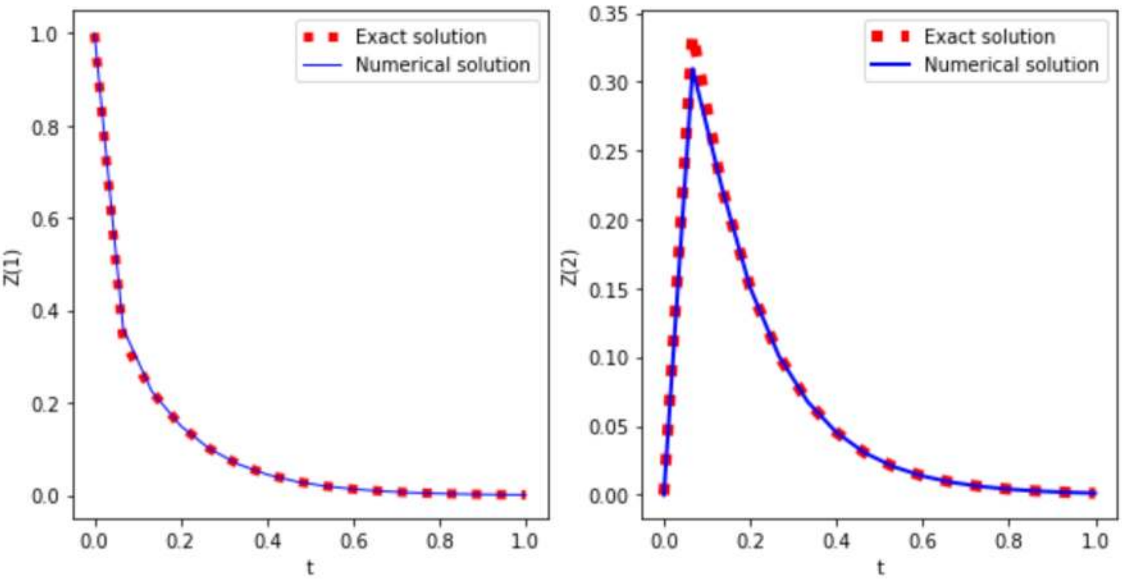


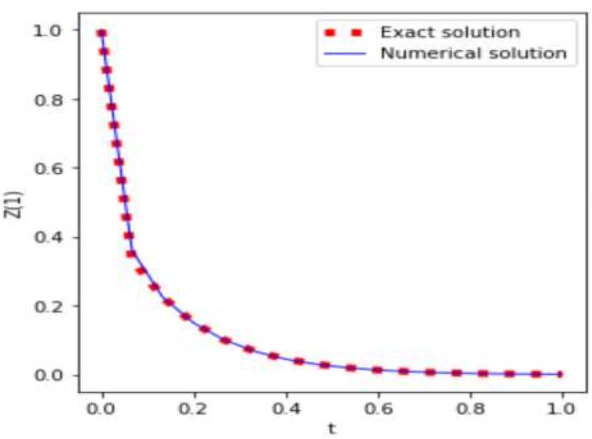
Figure 1: Profile solution for Problem 1

Table 2: Comparison Result of z_1 for Problem 1

t	$z(t)$	$z_1(t)$	$\lvert z(t) - z_1(t)\rvert$
0.20	0.33530156446464362999	0.06767200273471479024 7	2.54284×10^{-2}
0.40	0.22466441197379730427	0.22469925984189926044	3.48478×10^{-5}
0.60	0.15059710594701431165	0.15059886462150065052	1.75860×10^{-6}
0.80	0.10094825899733647829	0.10097264863920704481	2.43896×10^{-5}
1.00	0.06766764161830634611	0.067672002734714790247	4.36116×10^{-6}
1.20	0.04535897664470625168	0.04536118382991449059 3	2.20718×10^{-6}
1.40	0.03040503131260898249	0.03040650981639935594 2	1.47850×10^{-7}
1.60	0.02038110198918310758	0.02038209418731861997 1	9.92198×10^{-7}
1.80	0.01366186122364628040	0.01366284629344548635 3	9.85069×10^{-7}
2.00	0.00915781944436709014	0.00915848749567696864 0	6.68051×10^{-7}
2.20	0.00613866995153422058	0.0061391177562605422358	4.47804×10^{-7}
2.40	0.00411487352451001442	0.0041151737191594553502	3.00194×10^{-7}
2.60	0.00275828221038038621	0.0027585481612003116988	2.65950×10^{-7}
2.80	0.00184893185824146541	0.0018491116996924581333	1.79841×10^{-7}
3.00	0.00123937608833317921	0.0012394966389739651914	1.20550×10^{-7}

Table 3: Comparison Result of z_2 for Problem 1

t	$z(t)$	$z_2(t)$	$\lvert z(t) - z_2(t)\rvert$
0.20	0.33535037428834497180	0.30960572256641213744	2.57446×10^{-2}
0.40	0.22466451974417424401	0.22464062855752519152	2.38911×10^{-5}
0.60	0.15059710593100098342	0.15060266988864076734	5.56395×10^{-6}
0.80	0.10094825899732591356	0.10092877892490726787	1.94800×10^{-6}
1.00	0.06766764161830634894	0.06766974703465251893	2.10541×10^{-6}
1.20	0.04535897664470625168	0.04536118109942907980	2.20445×10^{-6}
1.40	0.03040503131260898249	0.03040650998570001563	1.47867×10^{-7}
1.60	0.02038110198918310758	0.02038209226510866320	9.90275×10^{-7}
1.80	0.01366186122364628040	0.01366284619401490224	9.84970×10^{-7}
2.00	0.00915781944436709014	0.00915848749555003001	6.68051×10^{-7}
2.20	0.00613866995153422058	0.00613911775626806871	4.47804×10^{-7}
2.40	0.00411487352451001442	0.00411517371907533865	3.00194×10^{-7}
2.60	0.00275828221038038620	0.00275854816119593290	2.65950×10^{-7}
2.80	0.00184893185824146541	0.00184911169969245197	1.79841×10^{-7}
3.00	0.00123937608833317921	0.00123949663897396534	1.20550×10^{-7}



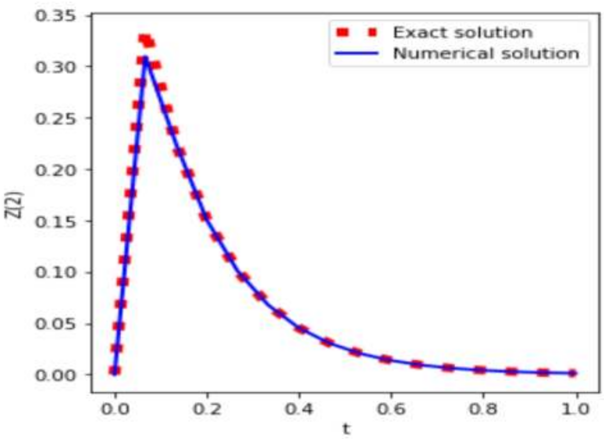


Figure 2: Profile solution for Problem 2

Table 4: Comparison Result of z_1 for Problem 2

t	$z(t)$	$z_1(t)$	$\lvert z(t) - z_1(t) \rvert$
0.0625	1.7812388434267357035714	1.7224741201493165655975	5.87647×10^{-2}
0.1250	1.5741655206295851644515	1.5735784574373308188485	5.87063×10^{-7}
0.1875	1.3892017181724113003316	1.3892024037603170774055	6.85587×10^{-7}
0.2500	1.2259662270401725672575	1.2259388994057635266342	2.73276×10^{-7}
0.3125	1.0819113980702038580181	1.0819097671638695472152	1.63090×10^{-9}
0.3750	0.9547834576680082137107	0.9547834503875126024516	7.28049×10^{-9}
0.4375	0.8425934440310276216596	0.8425934516379089236209	7.60688×10^{-9}
0.5000	0.7435861044954685223724	0.7435861104615844738720	5.96611×10^{-9}
0.5625	0.6562124340221962623557	0.6562124427626918203525	8.74049×10^{-9}
0.6250	0.5791054404620863729971	0.5791054482856815168921	7.82359×10^{-9}
0.6875	0.5110587574776790508945	0.5110587643823243132580	6.90464×10^{-9}
0.7500	0.4510077705127836967800	0.4510077766062636108881	6.09347×10^{-9}
0.8125	0.3980129605191156331026	0.3980129676396217252892	7.12050×10^{9}
0.8750	0.3512452048466444049929	0.3512452111740265736755	6.32738×10^{-9}
0.9375	0.3099728053248554045335	0.3099728109087331320470	5.58387×10^{-9}
1.0000	0.2735500405846426751048	0.2735500455125017088059	4.92785×10^{-9}

Table 5: Comparison Result of z_2 for Problem 2

t	$z(t)$	$z_2(t)$	$\lvert z(t) - z_2(t) \rvert$
0.0625	-0.016245038257544897841	0.0425196927492879193219	5.87647×10^{-4}
0.1250	-0.016563954486775427961	-0.015976884280185252955	5.70702×10^{-6}
0.1875	-0.014623160590466903241	-0.014623839988320409986	9.75732×10^{-7}
0.2500	-0.012904907614905720049	-0.012877574517332491221	2.73330×10^{-7}
0.3125	-0.011388541032223374104	-0.011386900616537189755	1.64041×10^{-8}
0.3750	-0.010050352185978799434	-0.010050336396356883139	1.57896×10^{-11}
0.4375	-0.008869404674010816489	-0.008869404771661439535	9.76506×10^{-10}
0.5000	-0.007827222152583879181	-0.007827221491546657376	6.61037×10^{-11}
0.5625	-0.006907499305496802761	-0.006907499354057801868	4.85609×10^{-11}
0.6250	-0.006095846741706172347	-0.006095846823639266986	8.19330×10^{-11}
0.6875	-0.005379565868186095272	-0.005379565940867042176	7.26809×10^{-11}
0.7500	-0.004747450215924038913	-0.004747450280046560113	6.41225×10^{-11}
0.8125	-0.004189610110727532980	-0.004189610185679065957	7.49515×10^{-11}
0.8750	-0.003697317945754151631	-0.003697318012358163204	6.66040×10^{-11}
0.9375	-0.003262871634998477942	-0.003262871693776138244	5.87776×10^{-11}
1.0000	-0.002879474111417291316	-0.002879474163289491153	5.18721×10^{-11}

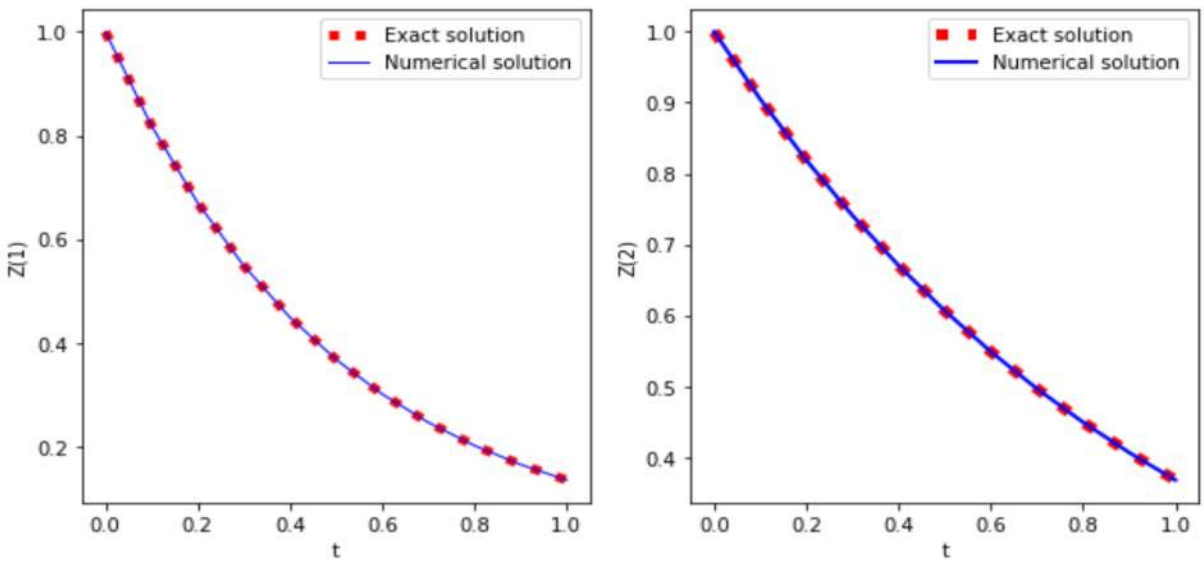


Figure 3: Profile solution for Problem 3

Table 6: Comparison Result of z_1 for Problem 3

t	$Z(t)$	$z_1(t)$	$\lvert Z(t) - z_1(t)\rvert$
0.100	0.1353352832366126918	0.135335283900553600380	6.63940×10^{-10}
0.200	0.0183156388887341802	0.018315639037978913619	1.49244×10^{-10}
0.300	0.0024787521766663584	0.002478752208879943511	3.22135×10^{-11}
0.400	0.0003354626279025118	0.000335462633345306306	5.44279×10^{-12}
0.500	0.0000453999297624848	0.000045399930718803095	9.56318×10^{-13}
0.600	$6.144212353328209 \times 10^{-6}$	$6.14421250257915187 \times 10^{-6}$	1.49250×10^{-13}
0.700	$8.315287191035678 \times 10^{-7}$	$8.31528743325846039 \times 10^{-7}$	2.42222×10^{-14}
0.800	$1.125351747192591 \times 10^{-7}$	$1.12535178360510108 \times 10^{-7}$	3.64125×10^{-15}
0.900	$1.522997974471262 \times 10^{-8}$	$1.52299803111905810 \times 10^{-8}$	5.66477×10^{-16}
1.000	$2.061153622438557 \times 10^{-9}$	$2.06115370575375547 \times 10^{-9}$	8.33151×10^{-17}

Table 7: Comparison Result of z_2 for Problem 3

t	$Z(t)$	$z_2(t)$	$\lvert Z(t) - z_2(t)\rvert$
0.100	0.3678794411714423216	0.3678794420738215812 9	9.02379×10^{-10}
0.200	0.1353352832366126918 9	0.1353352837879586940 3	5.51346×10^{-10}
0.300	0.0497870683678639429 79	0.0497870686913761658 30	3.23512×10^{-10}
0.400	0.0183156388887341802 94	0.0183156390373116348 28	1.48577×10^{-11}
0.500	0.0067379469990854670 96	0.0067379470700503895 61	7.09649×10^{-11}
0.600	0.0024787521766663584 23	0.0024787522067716371 36	3.01052×10^{-11}
0.700	0.0009118819655545162 08	0.0009118819788359680 39	1.32814×10^{-12}
0.800	0.0003354626279025118 38	0.0003354626333296136 93	5.42710×10^{-12}
0.900	0.0001234098040866795 49	0.0001234098063817853 72	2.29510×10^{-13}
1.000	0.0000453999297624848 51	0.0000453999306800401 42	9.17555×10^{-13}

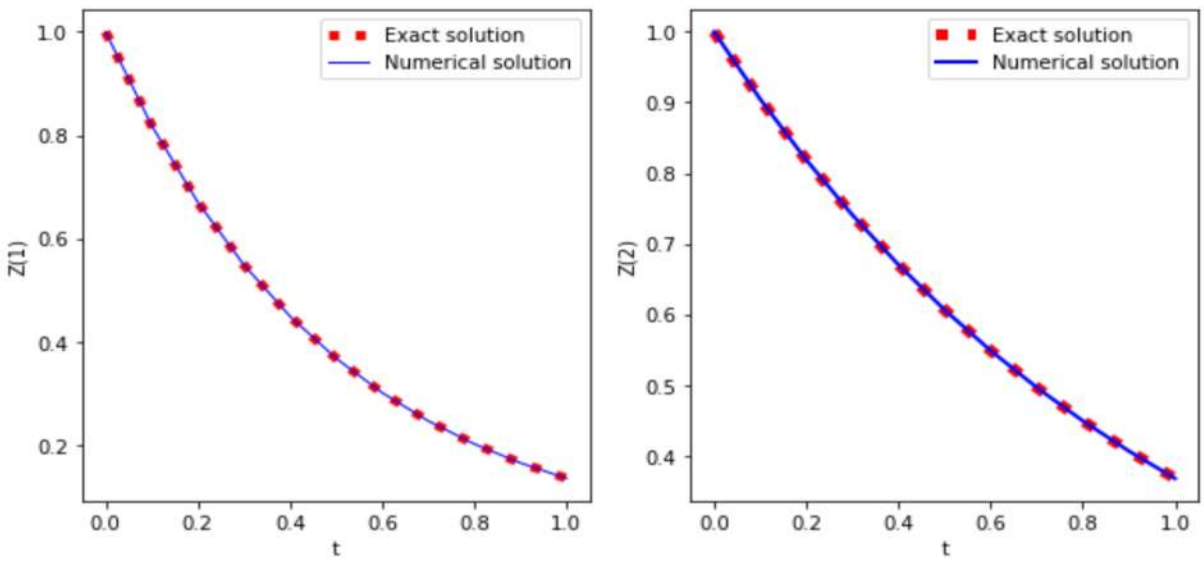


Figure 4: Profile solution for Problem 4

Table 8: Comparison Result of z_1 for Problem 4

t	$Z(t)$	$z_1(t)$	$\lvert Z(t) - z_1(t)\rvert$
0.100	0.81873075307798185867	0.81873075717909515030	4.10111×10^{-9}
0.200	0.67032004603563930074	0.67032004732826214578	1.29262×10^{-9}
0.300	0.54881163609402643263	0.54881163715192314629	1.05789×10^{-9}
0.400	0.44932896411722159143	0.44932896498600046686	8.68778×10^{-9}
0.500	0.36787944117144232160	0.36787944368730752857	2.51586×10^{-9}
0.600	0.30119421191220209664	0.30119421304245564822	1.13025×10^{-10}
0.700	0.24659696394160647694	0.24659696486679413398	9.25187×10^{-10}
0.800	0.20189651799465540849	0.20189651875332366913	7.58668×10^{-10}
0.900	0.16529888822158653830	0.16529888964265752646	1.42107×10^{-9}
1.000	0.13533528323661269189	0.13533528398193333375	7.45320×10^{-10}

Table 9: Comparison Result of z_2 for Problem 4

t	$Z(t)$	$z_2(t)$	$\lvert Z(t) - z_2(t)\rvert$
0.100	0.90483741803595957316	0.90483741888208201073	8.46122×10^{-10}
0.200	0.81873075307798185867	0.81873075386712958296	7.89147×10^{-10}
0.300	0.74081822068171786607	0.74081822139576027818	7.14042×10^{-10}
0.400	0.67032004603563930074	0.67032004668177876646	6.46139×10^{-10}
0.500	0.60653065971263342360	0.60653066083442824593	1.12179×10^{-9}
0.600	0.54881163609402643263	0.54881163712358090997	1.02955×10^{-9}
0.700	0.49658530379140951470	0.49658530472298378291	9.31574×10^{-10}
0.800	0.44932896411722159143	0.44932896496017530462	8.42953×10^{-10}
0.900	0.40656965974059911188	0.40656966084988405816	1.10928×10^{-9}
1.000	0.36787944117144232160	0.36787944218432163944	6.63940×10^{-9}

Discussion of Results

The newly derived block Milne technique is applied to stiff initial value problems in ordinary differential equations of the first order. The present technique associates numerical results with their exact solutions and summarizes the results in graphs and tables. The graphs of the exact solutions versus the numerical solutions for problems 1 to 4 are presented in Figures 1 to 4, which demonstrate that the numerical results are in good agreement with the exact solutions. In addition, the absolute errors associated with the numerical results and the analytic solutions are compared in Tables 2–9. The relatively small difference between the exact answer and the computed results proves the validity of the derived technique.

Conclusion

Using the collocation methodology, we established a self-starting hybrid block Milne technique with a greater degree of accuracy in this study. The novel technique aims to increase the efficacy and precision of Linear Multistep techniques by increasing the number of steps at both grid and off-grid locations. In the creation of the new approach, four off-step points and four step points were selected. The convergence of the suggested technique's fundamental attributes was analyzed. Systems of stiff problems were solved numerically to illustrate the accuracy of the suggested technique. The numerical outcomes of the problems demonstrate the effectiveness of the proposed technique, as the computed outcomes corresponded well with the exact solutions. Based on the graphs and tabulated data, we can infer that the proposed technique is an appropriate alternative for dealing with stiff problems that exist in all disciplines of science and engineering. For all computations, Maple 2015 was utilized.

References

Amat, S., Legaz, M. J., & Ruiz-Álvarez, J. (2019). On a variational method for stiff differential equations arising from chemistry kinetics. *Mathematics, 7*(459), 1-11.

Aslam, M., Farman, M., Ahmad, H., Gia, T. N., Ahmad, A., & Askar, S. (2021). **Fractal fractional derivative on chemistry kinetics hires problem.** *AIMS Mathematics, 7*(1), 1155–1184. doi: 10.3934/math2022068.

Badmus, A. M., Yahaya, Y. A. & Pam, Y. C. (2015). Adams Type Hybrid Block Methods Associated with Chebyshev Polynomial for the Solution of Ordinary Differential Equations. *British Journal of Mathematics and Computer Science*, *6*(6), 464-474.

Chapra, S. C., & Canale, R. P. (2015). *Numerical Methods for Engineers*, Seventh Edition. McGraw-Hill Education, New York.

Ehiemua, M. E. & Agbeboh, G. U. (2019). On the Derivationof a new fifth-order Implicit Runge-Kutta Scheme for Stiff Problems in Ordinary Differential Equation. *Journal of Nigerian Mathematical Society*, *38* (2), 247-258.

Eziokwu, C. E. & Okereke, N. R. (2020). On Euler's and Milne's linear multi step methods of solving the Ordinary Differential Equations. *Asian Journal of Pure and Applied Mathematics*, *2*(2), 1-18.

Garba, J. & Mohammed, U. (2020). Derivation of a New One-Step Numerical Integrator for Solving First Order Ordinary Differential Equations. *Nigerian Journal of Mathematics and Applications,* 30, 155-172.

Gomathi, V. & Rabiyabe, G. (2022). Numerical solution of system of ordinary differential equations using Runge- Kutta method. *International Journal of Mechanical Engineering. 7*(4), 1444 – 1453.

Iyorter, B. V., Luga, T. & Isah, S. S. (2019). Continuous Implicit Linear Multistep Methods for the Solution of Initial Value Problems of First-Order Ordinary Differential Equations. *IOSR Journal of Mathematics, 15*(6), 51-64.

Kashkaria, B. S. H. & Syam, M. I. (2019). Optimization of one step block method with three hybrid points for solving first-order ordinary differential equations. *Results in Physics*, 12, 592-596. Doi:10.1016/j.rinp.2018.12.015

Khalsaraei, M. M., Shokri, A., & Molayi, M. (2020). The new class of multistep multiderivative hybrid methods for the numerical solution of chemical stiff systems of first order IVPs. *Journal of Mathematical Chemistry*. https://doi.org/10.1007/s10910-020-01160-z.

Kwanamu, J. A., Skwame, Y. & Sabo, J. (2021). Block hybrid method for solving higher order ordinary differential equation using power series on implicit one-step second derivative. *FUW Trends in Science & Technology Journal*, *6*(2), 576 – 582.

Ma'ali, A. I., Mohammed, U., Audu, K. J., Yusuf, A., & Abubakar, A. D. (2020). Extended Block Hybrid Backward Differentiation Formula for Second Order Fuzzy Differential Equations using Legendre Polynomial as Basis Function. *Journal of Science, Technology, Mathematics and Education*, *16*(1), 100 -111.

Mazarina, M. N., & Syahirbanun, I. (2022). Milne-Simpson method for solving first order fuzzy differential equations using hukuhara approach. *Enhanced Knowledge in Sciences and Technology*, *2*(1), 508-516.

Mohammed, U., Garba, J. & Semenov, M. E. (2021). One-step second derivative block intra-step method for stiff system of ordinary differential equations. Journal of Nigerian Mathematical Society *40*(1), 47-57.

Ndipmong, A.U. & Udechukwu, P. E. (2022). On the analysis of numerical methods for solving first order non-linear ordinary differential equations. *Asian Journal of Pure and Applied Mathematics*, *4*(3), 279-289.

Oghonyon, J. G., Agboola, O. O., Adesanya, A. O., & Ogunniyi, O. P. (2018a). Computing oscillating vibrations employing exponentially fitted block Milne's device. *International Journal of Mechanical Engineering and Technology*, *9*(8), 1234–1243.

Oghonyon, J.G., Ehigie, J. & Eke, S. K. (2016a). Investigating the convergence of some selected properties on block predictor-corrector methods and it's applications. *Journal of Engineering and Applied Sciences,* 11, 2402-2408.

Oghonyon, J. G., Okunuga, S. A**.,** Eke, K. S., & Odetunmibi, O. A. ***(2018b).*** Block Milne's Implementation for Solving Fourth Order Ordinary Differential Equations. ***Engineering, Technology and Applied Science Research, 8(3), 2943-2948.*** doi: 10.48084/etasr.1914.

Oghonyon, J. G., Okunuga, S. A., & Iyase, S. A. (2016b). **Milne's Implementation on block Predictor-corrector methods.** ***Journal of Applied Sciences,*** *16*(5), 236-241. doi: 10.3923/jas.2016.236.241.

Tiamiyu, A. T., Cole, A. T., & Audu, K. J. (2021). A Backward Differentiation Formula for Third Order Initial or Boundary Values Problems Using Collocation Method. *Iranian Journal of Optimization*, *13*(2), 81-92.

APPLICATION OF PULTZER'S ALGORITHM TO FRACTIONAL DIFFERENTIAL EQUATION MODEL OF DECAY-GROWTH OF RADIOACTIVE SUBSTANCES

[1]Modebei M. I., [1]Olaiya O. O.[2]Owolanke A. O.
[1]Mathematics Programme, National Mathematical Centre,
Abuja, Nigeria.
[2]Department of Mathematics Olusegun Agagu University of Science and Technology,
Okitipupa, Ondo State, Nigeria.
Email: gmarc.ify@gmail.com

Abstract

Radioactive substances decay and as a result they create other substances which also have the affinity to decay. An example is Radium which decays to Radon and thus further decays to Polonium. In real sense, these decays take place in fractions. Suppose that initially a sample of pure radium decays. It is of interest to determine how much radium and radon the sample contain at time t. In this paper Pultzer algorithm for exponential function is appllied to make a close comparison of the integer order and fractional order derivatives using the decay-growth model equation of a radioactive substance thus giving the amount of sample it contains after disintegration at time t.

Keywords: Fractional derivatives, Fractional Differential Equation, Matrix exponential function, Mittag-Laffler function.

Introduction

Fractional derivative have been extensively investigated due to their broad applications in mathematics, physics and engineering [1,2], such as anomalous transport in disordered systems, some percolations in porous media, and the diffusion of biological populations. But these nonlinear fractional differential equations are difficult to get their exact solutions.

Rigorous mathematical theory has been developed. Geometrical interpretation or physical meaning exists. But it is not as straight forward as for the integer-order derivatives [3].

A typical example of differential equations involving fractional derivatives is the Bagley-Torvik equation of oscillatory processes with fractional damping [4]:

$$\frac{d^2}{dx^2}f(x) + \alpha D_t^{\frac{3}{2}}f(x) + \beta f(x) = g(x) \tag{1}$$

Existence and uniqueness of solutions of fractional differential eqauations have been established, analytical solutions are difficult to evaluate. Dedicated, elegant numerical methods exist as well [5].

Some of the merits of fractional derivatives are [2]:

1. Calculating time-fractional derivative of a function $f(t$ at some time $t = t_1$ which requires all the past history.
2. Employing fractional derivatives for modeling systems with memory.
3. Calculating space-fractional derivative of a function $f(x)\ at\ x = x_1$which requires all non-local $f(x)$ values.
4. Employing fractional derivatives for modeling distributed parameter systems.

Numerical methods have been employed in most cases for the solutions of most fractional differential equation as can be seen in [6,7].

In this paper we illustrate the rate at which a radioactive substance disintegrates with time fractionally as compared to the disintegration when modeled using the conventional differential equation with integer order using the Pultzer's algorithms to find the matrix exponential for a system of fractional differential equation.

Preliminaries

Gamma Function

This is otherwise refered to as the generalization of the factorial for all real numbers, defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x \in R \tag{2}$$

with the property that

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x) = (x-1)!$$

Mittag-Leffler Function

This function is a direct generalization of the conventional exponential function e^x, and it play a very important role in fractional calculus. The Mittag-Leffler function is defined in terms of power series as follows

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0 \tag{3}$$

For some particular values of α and β, the Mittage-Leffler function reduces to some farmiliar functions. For example

$$E_{1,1}(x) = e^x$$

$$E_{1,2}(x) = \frac{e^x - 1}{x}$$

Riemann-Liouville Fractional Integral

Definition 2.1 [8] *Let v be a non negative number, f be piecewise continuous on $I = (0,\infty)$ and integrable on any finite subinterval of $J = [0,\infty]$. Then for $t > 0$ we define the Riemann-Liouville fractional integral of f of order v as*

$$_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t) dt, \quad v > 0 \tag{4}$$

This definition can be obtained in several ways but we shall consider the linear differential equation.We first state a lemma.

Lemma 2.2 (Replacement Lemma) [8] *Suppose that $f : [a,b] \to R$ is continuous, then*

$$\int_a^x \int_a^{x'} f(t) dt dx' = \int_a^x (x-t) f(t) dt, \quad x \in [a,b]$$

Proof. See **[8]**

Abacus (Mathematics Science Series)
Vol. 50, No 1, April, 2023

Riemann-Liouville Fractional Derivative

The Riemann-Liouville fractional derivative can be defined using the definition of the fractional integral.

Definition 2.3 [8, 9] *let* $v = n - u$ *where* $0 < v < 1$ *and* n *is the smallest integer greater than* u. *The fractional derivative of* $f(x)$ *of order* u *is*

$$D^u f(x) = D^n[D^{-v} f(x)] \qquad \{5\}$$

If we wish to find the fractional derivative of $f(x) = x^\lambda$ of order v, where $\lambda \geq 0$, in order to use the definition above, we interchange u and v and let $u = n - v$ where $0 < u < 1$. Then we have that $n = 1$ and $u = 1 - v$, so that

$$\begin{aligned}
D^v f(x) &= D^1[D^{1-v} f(x)] \\
&= D^1[D^{1-v} x^\lambda] \\
&= D^1\left[\frac{\Gamma(\lambda+1)}{\Gamma((\lambda-v+1)+1)} x^{\lambda-v+1}\right] \\
&= \frac{(\lambda-v+1)\Gamma(\lambda+1)}{(\lambda-v+1)\Gamma(\lambda-v+1)} x^{\lambda-v}
\end{aligned}$$

$$D^v f(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)} x^{\lambda-v}$$

Fractional Differential Equation and Laplace Transform.

Here we shall apply Laplace transform to solve some fractional differential equation. This procedure ia analogous to solving the conventional differential equation with integer order using Laplace transform.

Definition 3.1 (Laplace Transform.) [8, 9]. *Let* $f(x)$ *be given for* $t \geq 0$ *and suppose that* $f(x)$ *is defined on some interval* I. *The Laplace transform of* $f(t)$ *denoted by* $\mathrm{L}\{f(t)\}$ *or equivalently* $F(s)$ *is defined by the equation*

$$\mathrm{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (6)$$

Theorem 3.2 *Suppose that*

(I) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A,

(II) $|f(t)| \leq ke^{st}$ where $t \geq M$, where k, a, M are real constants and $k, M > 0$.

Then, $\mathrm{L}\{f(t)\} = F(s)$ defined in (6) above exist for $s > a$.

Such functions are described as piecewise continuous, and of exponential order as $t \to \infty$.

$\mathrm{L}^{-1}\{f(t)\}$ is the inverse Laplace transform of $f(t)$ and is unique.

The Laplace transform is a Linear operator. that is closed under addition and scalar multiplication. The laplace transform of t^p where $p > -1$ is given by

$$\mathrm{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx = \frac{\Gamma(p+1)}{s^{p+1}}$$

Abacus (Mathematics Science Series)
Vol. 50, No 1, April, 2023

and

$$\mathcal{L}\{e^{\alpha t}\} = \int_0^\infty e^{(\alpha-s)t}\,dt = \frac{1}{s-\alpha}$$

Laplace Transform of Fractional Integral

The Laplace transform of $f(t)$ of differential order v is given as

$$\mathcal{L}\{D^{-v} f(t)\} = \frac{1}{\Gamma(v)}\mathcal{L}\{t^{v-1}\}\mathcal{L}\{f(t)\} = s^{-v}F(s) \qquad (7)$$

which is the Laplace transform of (4).

Laplace Transform of Fractional Derivative

The Laplace transform of integer order operator of $f^{(n)}$ is given as

$$\begin{aligned}\mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\\ &= s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)\end{aligned}$$

we thus recall from (5) and noting that $u = n - v$, then

$$D^v f(t) = D^n[D^{-(n-v)} f(t)]$$

Assume that $\mathcal{L}\{f(t)\}$ exist, then

$$\begin{aligned}\mathcal{L}\{D^v f(t)\} &= \mathcal{L}\{D^n[D^{-(n-v)} f(t)]\}\\ &= s^n\mathcal{L}\{D^n[D^{-(n-v)} f(t)]\} - \sum_{k=0}^{n-1} s^{n-k-1} D^k[D^{-(n-v)} f(t)]|_{t=0}\\ &= s^n[s^{-(n-v)}F(s)] - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-v)} f(0)\\ &= s^v F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-v)} f(0)\end{aligned}$$

More precisely. if we take $n = 2$ we obtain

$$\mathcal{L}\{D^v f(t)\} = s^v F(s) - sD^{-(2-v)} f(0) - D^{-(1-v)} f(0), \quad 0 < v \le 2$$

To illustrate this we thus solve the fractional differential equation $D^{1/2} f(t) = \alpha f(t)$.
Finding the Laplace transform of both sides we obtain

$$s^{1/2}F(s) - D^{-1/2} f(0) = \alpha F(s)$$

It is quite easy to see that $D^{-1/2} f(0)$ is a constant say λ, so that we have

$$s^{1/2}F(s) - \lambda = \alpha F(s)$$

$$F(s) = \frac{\lambda}{s^{1/2} - \alpha}$$

Finding the inverse Laplace transform we finally obtain

$$f(t) = \mathrm{L}^{-1}\left\{ \frac{\lambda}{s^{1/2} - \alpha} \right\} = \lambda t^{-1/2} E_{1/2,1/2}(\alpha t^{1/2})$$

Main Result

We consider the case where the decay of a sample A is the growth of another sample B and the dacay of a sample B is the growth of another sample C as follows. Suppose that x_1 at time $t = 0$ denoted by $x_1(0)$ is the initial mass of a sample A i.e $x_0(0) = m_0$

$$A(= x_1(t)) \xrightarrow{decay\ constant\ a} \quad B(= x_2(t)) \xrightarrow{decay\ constant\ b} \quad C(= x_3(t)) \xrightarrow{decay\ constant\ c} \cdots$$

with the following dynamics

$$\frac{dx_1(t)}{dt} = -ax_1(t) \tag{8}$$

$$\frac{dx_2(t)}{dt} = ax_1(t) - bx_2(t) \tag{9}$$

$$\frac{dx_3(t)}{dt} = bx_2(t) - cx_3(t) \tag{10}$$

$$x_1(0) = m_0,\ x_2(0) = x_3(0) = 0 \tag{11}$$

Notice that (8) is a decay process while (9) and (10) are both decay and growth processes. Thus, considering equation (8), (9) and (10) we have the equivalent form as

$$x'(t) = Ax(t) \tag{12}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, A = \begin{pmatrix} -\alpha & 0 & 0 \\ \alpha & -\beta & 0 \\ 0 & \beta & -\gamma \end{pmatrix}, x(0) = \begin{pmatrix} m_0 \\ 0 \\ 0 \end{pmatrix}$$

In fractional order we have

$$D^{v}x(t) = Ax(t), \quad D^{-(1-v)}x(0) = \delta, \quad v \in (0,1) \tag{13}$$

If$v \equiv 1$, the system (13) becomes (12). In this work we make a very close comparison between the systems (12) and (13) as $v \to 1$.

Theorem 4.1 *If matrixA* $\in R^{n\times n}$ *(or* $C^{n\times n}$*) has a distinct eigenvalues* $\lambda_i \neq \lambda_j,\ i,j = 1,2,\dots,m$, *then it has (at least)* m *linearly independent eigenvectors.*

Corollary 4.2 *If all eigenvalues of A are distinct then A is diagonalizable, that is there exist an invertible matrixP and a diagonal matrix* $D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ *such that* $A = PDP^{-1}$.

Now to determine the exponential of a matrix A we may employ Corollary 4.2 that is

$$e^{At} = Pe^{Dt}P^{-1}$$

Recall that in this case $A = \begin{pmatrix} -\alpha & 0 & 0 \\ \alpha & -\beta & 0 \\ 0 & \beta & -\gamma \end{pmatrix}$ the eigenvalues are $\lambda_1 = -a,\ \lambda_2 = -b,\ \lambda_3 = -c$ so that

$$D = \begin{pmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix},\quad e^{Dt} = \begin{pmatrix} e^{-at} & 0 & 0 \\ 0 & e^{-bt} & 0 \\ 0 & 0 & e^{-ct} \end{pmatrix},$$

$$P = \begin{pmatrix} \dfrac{(a-b)(a-c)}{ab} & 0 & 0 \\ \dfrac{c-a}{b} & \dfrac{c-b}{b} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

So that by computation we have that

$$e^{At} = Pe^{Dt}P^{-1} =$$

$$\begin{pmatrix} -\dfrac{b\,e^{-at}(a-b)(a-c)}{-a^2b+ab^2+abc-cb^2} & 0 & 0 \\ \dfrac{a}{b}\,e^{-bt}\dfrac{b-c}{ab^2+b^2c+b^3-abc} + \dfrac{a}{b}\,e^{-at}\dfrac{a-c}{-a^2b+ab^2+abc-cb^2} & e^{-bt} & 0 \\ \dfrac{a\,e^{-ct}}{-b^2c+ab^2+bc^2-abc} - \dfrac{a\,e^{-at}}{-a^2b+ab^2+abc-cb^2} - \dfrac{a\,e^{-bt}}{ab^2+b^2c-b^3-abc} & \dfrac{b\,e^{-ct}}{b-c} - \dfrac{b\,e^{-bt}}{b-c} & e^{-ct} \end{pmatrix}$$

Hence the solution of system (12) becomes

$$\bar{x} = e^{At}x(0)$$

then obtaining

$$\bar{x} = \begin{pmatrix} m_0\left(-\dfrac{b\,e^{-at}(a-b)(a-c)}{-a^2b+ab^2+abc-cb^2}\right) \\ m_0\left(\dfrac{a}{b}\,e^{-bt}\dfrac{b-c}{ab^2+b^2c+b^3-abc} + \dfrac{a}{b}\,e^{-at}\dfrac{a-c}{-a^2b+ab^2+abc-cb^2}\right) \\ m_0\left(\dfrac{a\,e^{-ct}}{-b^2c+ab^2+bc^2-abc} - \dfrac{a\,e^{-at}}{-a^2b+ab^2+abc-cb^2} - \dfrac{a\,e^{-bt}}{ab^2+b^2c-b^3-abc}\right) \end{pmatrix}$$

We need to find the exponential matrix in other to solve system (13). We shall need the following results.

Theorem 4.3 *If A is a square matrix, then the solution of system (13) is given by*

$$\bar{x}_v(t) = K_{v,v}(At^v)C \quad (15)$$

where$K_{v,v}(At^v) = t^{v-1}E_{v,v}(At^v)$ and $E_{v,v}(At^v)$ is the Mittag-Laffler function and C in an $n \times 1$ matrix.

Proof. For proof see [7].

Theorem 4.4 *(Putzer's algorithm) If $\lambda_i, i = 1,2, \dots, n$ are eigenvalues of an $n \times n$matrixA, then*

$$K_{v,v}(At^v) = \sum_{k=0}^{n-1} p_{k-1}(t)M_k$$

where

$$M_0 = I, \qquad M_k = \prod_{i=1}^{k}(A - \lambda_i I), \;\; i \le k \le n-1.$$

and

$$D^v p_1(t) = \lambda_1 p_1(t), \quad D^v p_k(t) = \lambda_k p_k(t) + p_{k-1}(t)$$

With conditions

$$D^{-(1-v)}p_0(t) = 1, \quad D^{-(1-v)}p_k(t) = 0, \qquad 2 \le k \le n.$$

By Putzer's algorithm it can be easily seen that

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ a & a-b & 0 \\ 0 & 0 & c-a \end{pmatrix}$$
$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ab & b(a-c) & (c-a)(b-c) \end{pmatrix}$$

Solving

$$D^v p_1(t) = \lambda_1 p_1(t)$$
$$D^v p_2(t) = \lambda_2 p_2(t) + p_1(t)$$
$$D^v p_3(t) = \lambda_3 p_3(t) + p_2(t)$$

On solving we obtain

$$p_1(t) = X,$$
$$p_2(t) = \frac{X-Y}{k_{12}}$$

$$p_3(t) = \frac{Y-Z}{k_{12}k_{23}} - \frac{X-Z}{k_{12}k_{13}}$$

Where

$$X = t^{v-1}E_{v,v}(\lambda_1 t^v)$$
$$Y = t^{v-1}E_{v,v}(\lambda_2 t^v)$$
$$Z = t^{v-1}E_{v,v}(\lambda_3 t^v)$$
$$k_{12} = \lambda_2 - \lambda_1$$
$$k_{13} = \lambda_3 - \lambda_1$$
$$k_{23} = \lambda_3 - \lambda_2,$$
$$\lambda_1 = -a,\ \ \lambda_2 = -b,\ \ \lambda_3 = -c$$

But

$$K_{v,v}(At^v) = p_1(t)M_0 + p_2(t)M_1 + p_3(t)M_2$$

Estimating each term and simplifying we obtain\

$$K_{v,v}(At^v)$$

$$= \begin{pmatrix} X & 0 & 0 \\ \dfrac{a(X-Y)}{k_{12}} & X + \dfrac{(a-b)(X-Y)}{k_{12}} & 0 \\ ab\left[\dfrac{Y-Z}{k_{12}k_{23}} - \dfrac{X-Z}{k_{12}k_{13}}\right] & \dfrac{b(X-Y)}{k_{12}} + b(a+c)\left[\dfrac{Y-Z}{k_{12}k_{23}} - \dfrac{X-Z}{k_{12}k_{13}}\right] & \dfrac{(a+c)(X-Y)}{k_{12}} + (a+c)(b-c)\left[\dfrac{Y-Z}{k_{12}k_{23}} - \dfrac{X-Z}{k_{12}k_{13}}\right] \end{pmatrix}$$

By Theorem 4.3, the solution of system (13) where$\delta = \begin{pmatrix} m_0 \\ 0 \\ 0 \end{pmatrix}$ is given as

$$\vec{x}_v(t) = K_{v,v}(At^v) * \delta$$

$$\vec{x}_v(t) = \begin{pmatrix} m_0 X \\ m_0 \frac{a(X-Y)}{k_{12}} \\ m_0 ab\left[\frac{Y-Z}{k_{12}k_{23}} - \frac{X-Z}{k_{12}k_{13}}\right] \end{pmatrix}$$

Using the following constants $m_0 = 1620,\ \ a = 4.25 \times 10^{-4},\ \ b = 2.01 \times 10^{-6},\ \ c = 5.81 \times 10^{-8}$ (See [10,11]).

Ploting x and $\bar{x}_v$ in equations (14) and (16) respectively against time t yields Figure 1 which shows the amount of Radium decaying with time, respectively for $v = 0.90, 0.94, 0.96, 0.98, 0.99$.

Plotiing $y,\ y_v$, and $z,\ z_v$ and in equations (14) and (16) respectively against time t yields Figure 2 and Figure 3 which shows the amount of Radon and polonium growing with time, respectively for$v = 0.90, 0.94, 0.96, 0.98, 0.99$.

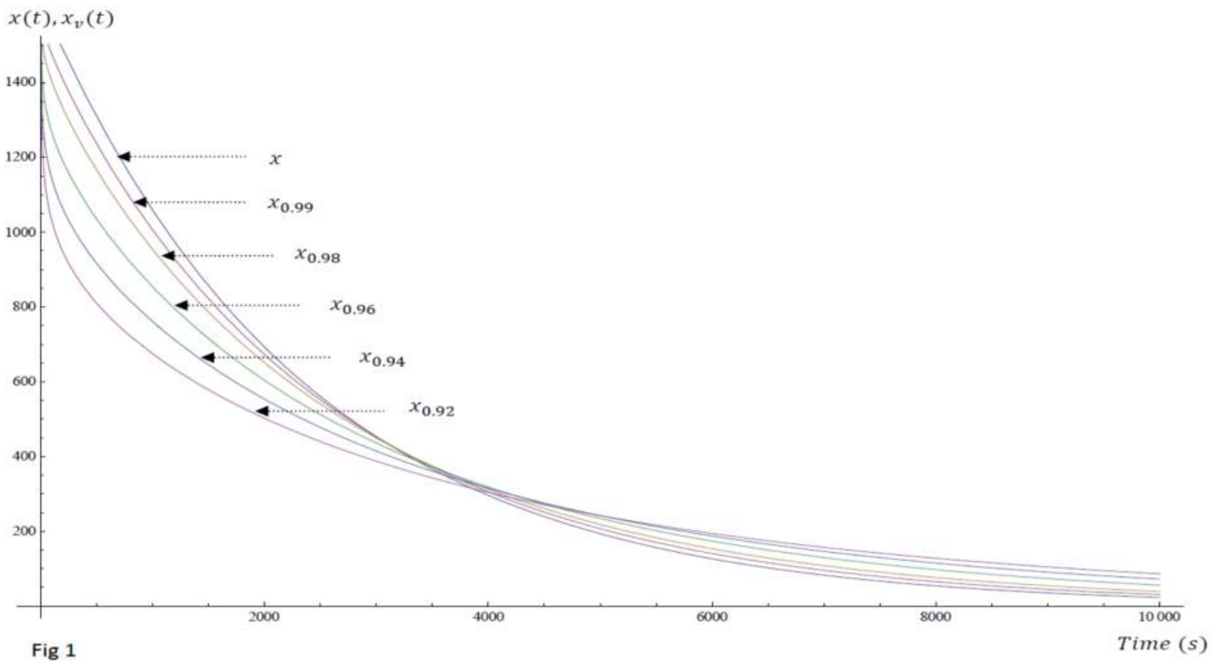


Fig 1

Figure 1. Comparison of the curves $x(t)$ and $x_v(t)$ i.e decay of Radium atom

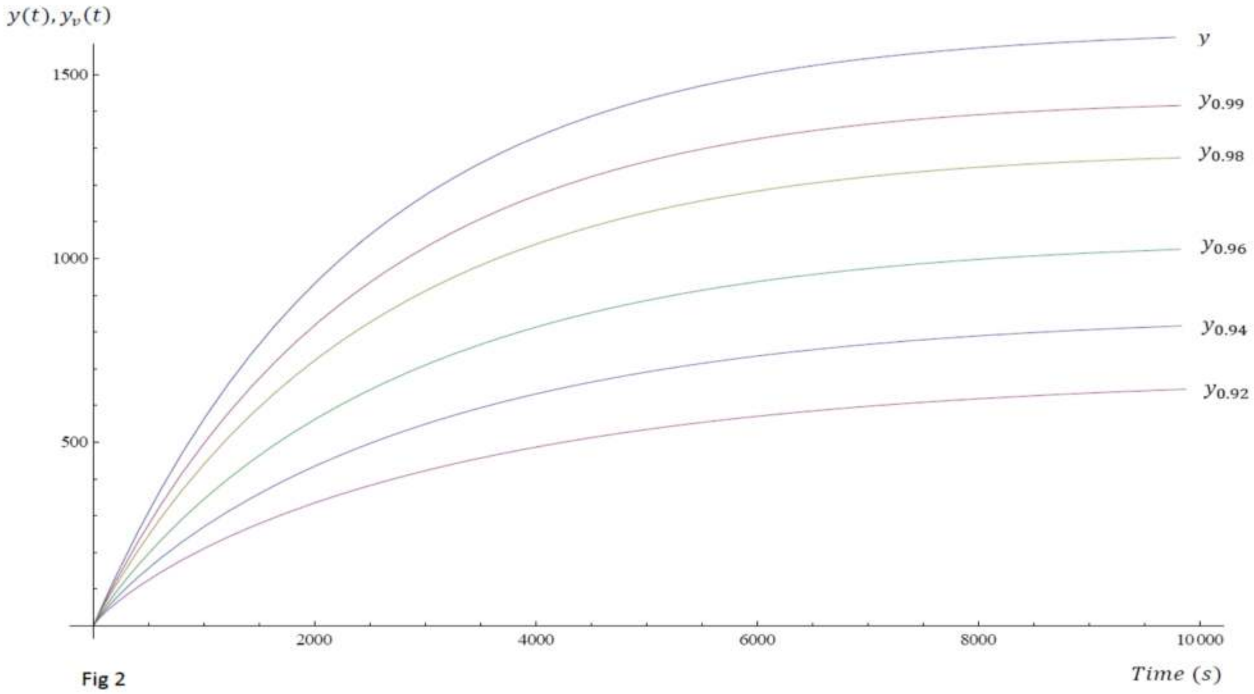


Fig 2

Figure 2. Comparison of the curves $y(t)$ and $y(t)$ i.e growth of Radon atom

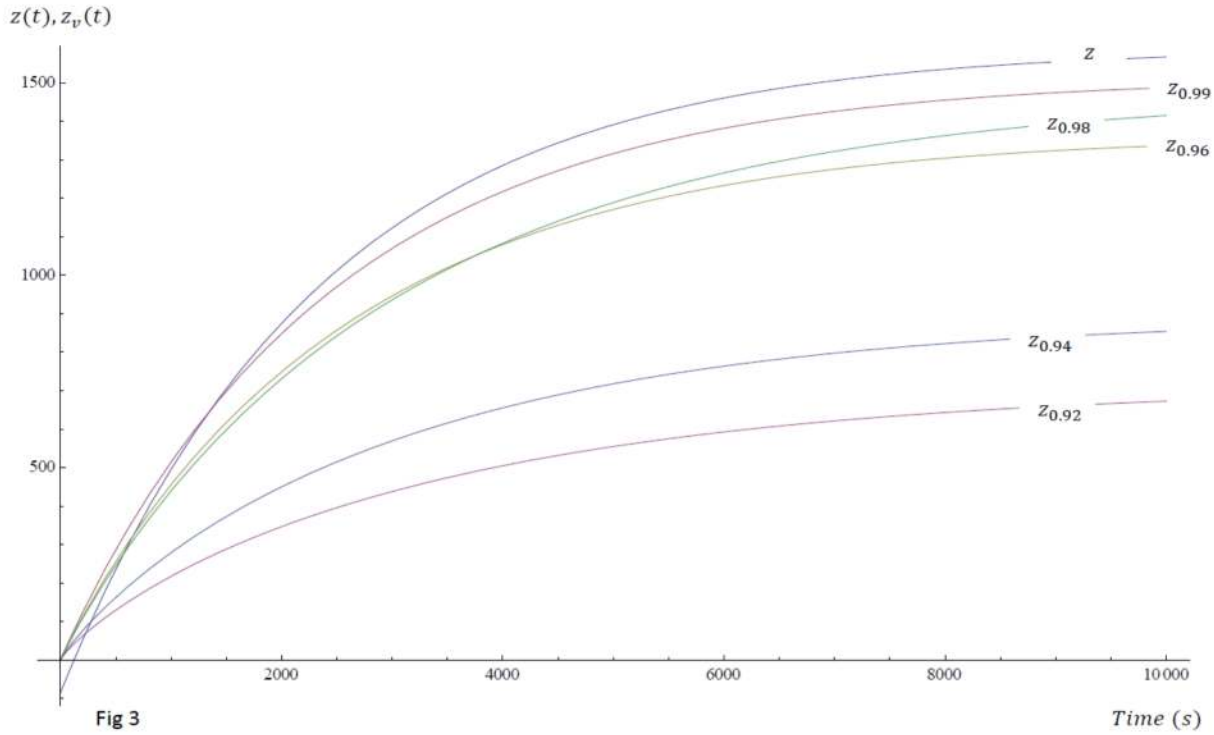


Figure 3. Comparison of the curves $z(t)$ and $z_v(t)$ i.e growth of Polonium atom.

It can be noticed from Figs. 1-3, that as $v \to 1$, $x_v, y_v, z_v \to x, y, z$ respectively, that is, the solutions of the fractional order approaches the solutions of the integer order.

Conclusion

From the analysis made so far, it can be seen that fractional orders tells precisely how the atom in this subject disintegrate fractionally and as well grow fractionally as this can be summed up to the approximated integer order case. As a matter of fact, fractional calculus has the potential of presenting intriguing and useful applications in the future. It is also important to note that a lot of researches is still on going in this direction.

References

Ross, B. (1975): Fractional Calculus and its Applications, Lecture Notes in Mathematics no. 457, Springer Verlag, Berlin. USA.

Samko S.G., Kilbas, A.A. and O.I. Marichev (1993): Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Amsterdam 1993.

Barret, J. H (1954).: Differential equations of non-integer order, Canad. J. Math., 6, 529-541.

Beyer, H. and S. Kempfle (1995): Definition of physically consistent damping laws with \ fractional derivatives, ZAMM, 75, 623-635.

Matignon D. (1966): Stability results for fractional differential equations with applications to control processing. Proceedings Computational Engineering in Systems and Application multiconference, IMACS, IEEE-SMC, Lille, France, July, 963-968.

Ramzi B. Albadarneh, Iqbal M. Batiha, (2016). Mohammad Zurigat, Numerical solutions for linear fractional differential equations of order 1 < α < 2 using finite difference method (FFDM), J. Math. Computer Sci. 16, 103-111

T. A. Biala and S. N. Jator (2015). Block Backward Differentiation Formulas for Fractional Differential Equations, International Journal of Engineering Mathematics 2015, Article ID 650425, 14 pages http://dx.doi.org/10.1155/2015/650425

Modebei M. I., Olaiya O. O. (2013). – Application of Fractional Differential Equation to Mortgage Problem. Inter Journ of App Math Resr IJAMR 2 (4). 505-511

Miller, K.S. and B. Ross (1993), An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York.

Kimeu, Joseph M. (2009)., "Fractional Calculus: Definitions and Applications" Masters Theses & Specialist Projects. Paper 115. http://digitalcommons.wku.edu/theses/115

Chisté V. M, Dulieu B. C. (2010).. Decay data evaluation of Radium – 226 and its daughters.) Laboratoire National Henri Becquerel Saclay 22, 94-110.

STABILITY ANALYSIS OF SOLUTIONS FOR A CLASS OF THIRD-ORDER NON LINEAR DUFFING-TYPE DIFFERENTIAL EQUATION

Kalu, Uchenna[1] and Anozie, V.O[2]
Department of Mathematics,
Michael Okpara University of Agriculture, Umudike
Abia State, Nigeria.
Email: uchennakalu304@gmail.com[1], anozievictorobinna@gmail.com[2]
Phone: +2347034689302 +2347035360703

Abstract

In this paper, the eigenvalue method is applied to study the stability of solutions for a class of third-order nonlinear Duffing-type differential equation. By dimensionalizing the equation to a first-order system, the nonlinear parts of each of the equivalent system derived is linearized using Maclaurin series expansion, stability solutions were investigated and it was concluded that since all the eigenvalues do not all have negative real parts, the system is unstable.

Keywords: Duffing-type ODE, eigenvalue method, Linearization, Stability.

Introduction

Several articles have appeared in the literature relating to the stability of periodic solutions of nonlinear differential equations of Duffing-type [1, 2, 3, and 4]. The existence of solutions of ordinary differential equation using implicit function theorem have been investigated by several notable authors [3, 4, and 5]. Other stability analysis of fractional Duffing oscillator used implicit function theorem to show the existence of periodic solutions for nonlinear partial differential equations [3, 7]. Other researchers investigated the stability of solutions of certain order types of delay differential equations [9-12], etc.

Consider the Duffing type equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = g(t) \tag{1.1}$$

where a, b, c are real constants and $g(t)$ is continuous, this has been used in engineering, economics, physics and many other physical phenomena. Given its characteristic oscillatory and chaotic nature many scientists are inspired by this nonlinear differential equation given its ability to replicate similar dynamics in our natural world. The nonlinear differential equation is used to model damped and derived oscillators [6]. These equations together with the Vander Pol's equation have become one of the most common examples in nonlinear oscillations. Due to the importance of the Duffing equation in real world problems, the study of existence of solution of the equation has continued to attract the attention of many researchers. The existence of solution of Duffing equation of the general form:

Abacus (Mathematics Science Series)
Vol. 50, No 1, April, 2023

$$\ddot{x} + c\dot{x} + g(t,x) = g(t) \tag{1.2}$$

where $g(t)$ is continuous and 2π-periodic in $t \in \mathbb{R}$, $g(t,x) = ax + bx^2 + \mathcal{B}x^3$ *and* $\mathcal{B} = 2$ represents a hard spring, carried out by [1].

The aim of this paper is to investigate the stability of solutions for a class of Third Order Nonlinear Differential equation of the Duffing-type

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = f(t) \tag{1.3}$$

where a, b, c, d are real constants and $f(t)$ is continuous using the eigenvalue method. The eigenvalues $\lambda_1, \ldots$ are obviously the roots of the polynomial.

Preliminaries

In order to reach our main results we will first give some important stability criteria for the general autonomous delay differential system.

Definition 2.1 (Stability)

Consider the nonlinear time-invariant system

$$\dot{x} = f(x), \quad \dot{f}: \mathbb{R} \to \mathbb{R}, \tag{2.2}$$

A point $x_e \in \mathbb{R}$, is an equilibrium point of the system if $f(x) = 0$.

We remark that x_e is an equilibrium point if and only if $x(t) = x_e$ is a trajectory.

Suppose x_e is an equilibrium point, then

i. The system is globally asymptotically stable if for every trajectory x, we have $x(t) \to x_e$ as t→ ∞.

ii. The system is locally asymptotically stable near or at x_e if there is an $\mathcal{R} > 0 \ni \| x(0) - x_e \| \leq \mathcal{R} \quad \Rightarrow x(t) \longrightarrow x_e \; as \; t \longrightarrow \infty$.

Definition 2.2

Consider also the linear system

$$\dot{x} = Ax \tag{2.4}$$

i. The system is said to be globally asymptotically stable with $\varkappa = 0$ if and only if $\Re_e\lambda_i(A) < 0, i = 1,2,3, \ldots n$, where $\Re_e$ *means the real part.*

ii. The system is said to be locally asymptotically stable (near $\varkappa = 0$) if and only if $\Re_e\lambda_i(A) < 0$,
$i = 1,2,3, \ldots n. Thus\ for\ linear\ system\ , Locally\ asymptoticaally\ stable \Leftrightarrow Globally\ asymptotically\ stable$).

Definition 2.3 (Asymptotic stability)

The equilibrium point $\varkappa$ is said to be asymptotically stable, if for all $\varepsilon > 0, \exists\, \delta > 0\ \ni$

i. $f(t; 0, \bar{x}) \in \beta(\varkappa, \varepsilon)$ for all$t \geq 0$.

ii. $\lim_{t\to\infty} f(t, 0, \bar{x}) = \varkappa$

Definition 2.4 (Lyapunov function)

Consider the differential equation

$$\dot{x} = f(x), \qquad f(0) = 0$$

Where the solutions are unique and vary continuously with the initial data. Let $V: \mathbb{R}^n \to \mathbb{R}$ be continuous together with its first partial derivatives, $\frac{\partial v}{\partial x}$ $(i = 1,2, \dots)$ on some open set $\Omega \subset \mathbb{R}^n$, where $\Omega = B_r(0)$, $\Omega = \{x: x\epsilon\mathbb{R}^n: \| x \| < r, where\ r\ is\ the\ radius\ of\ the\ ball\}$

i. A Function $V: \Omega \to \mathbb{R}$ is said to be positive definite/negative definite if $v(0) = 0\ and\ v$ assumes positive/negative values on Ω.

ii. A Function $V: \Omega \to \mathbb{R}$ is said to be positive/negative semi definite if $v(0) = (0)\ and\ v(x) \geq 0$ or $v(x) \leq 0$ on Ω.

If the function assumes arbitrary values, then it is said to be indefinite.

Theorem 1

The critical point $0 \in \mathbb{R}^n$ for the linear system $\dot{x} = A\underline{x}$ is asymptotically stable provided that all the eigenvalues of A have negative real parts otherwise it is unstable.

Theorem 2

Consider $\dot{x} = f(x)$, and assume that $f(0) = 0$

Linearization

$$\dot{x} = A(\underline{x}) + g(\underline{x}), \quad \| g(x) \| = 0(\| \underline{x} \|)\ as\ x \to 0$$

i. $\Re_e \lambda_k(A) < 0$, for $k \Rightarrow x = 0\ is$ Locally Asymptotically Stable $(L.A.S)$.

ii. $\exists\, K: \Re_e \lambda_k(A) > 0\ \Rightarrow x = 0\ is\ unstable.$

Definition 2.5 (Higher Order ODE and Reduction to the first Order System)

A general ODE of the order n resolved with respect to the highest derivative can be written in the form:

$$y^{(n)} = F\left(t, y, y', \dots, y^{(n-1)}\right) \tag{2.1}$$

Where t is an independent variable and $y(t)$ is an unknown function. It is sometimes more convenient to replace this equation by a system of ODEs of the first order. And any scalar ODE

can be represented by any system of first order differential equation. In practice, the system form is widely used in problems connected with stability, boundedness and periodicity of solutions.
Let $x(t)$ be a vector function of a real variable t, which takes values in $\mathbb{R}^n$. Denote by x_k the components of x, then the derivative $x'(t)$ is defined component-wise by $x'(x_1', x_2', \dots, x_n')$.

Now, consider a vector ODE of the first order

$$x' = f(t, x) \tag{2.2}$$

Where f is a given function of $n+1$ variables, which takes values in $\mathbb{R}^n$, that is $f: \Omega \to \mathbb{R}^n$, where Ω is an open subset of $\mathbb{R}^{n+1}$. Denoting by f_k the components of f, we can rewrite the vector equation (2.2) as a system of n scalar equations:

$$\left.\begin{array}{l} x_1' = f_1(t, x_1, \dots, x_n) \\ \dots \\ x_k' = f_1(t, x_1, \dots, x_n) \\ \dots \\ x_n' = f_1(t, x_1, \dots, x_n) \end{array}\right\} \tag{2.3}$$

A system of ODEs of the form (2.3) is called the normal system. This can be shown how (2.1) can be reduced to (2.3).
Suppose we associate the vector function $x = \left(y, y', \dots, y^{(n-1)}\right)$, which takes in $\mathbb{R}^n$. That is

$$x_1 = y, x_2 = y', \dots, x_n = y^{(n-1)} \Rightarrow x' = \left(y', y'', \dots, y^{(n)}\right) \tag{2.4}$$

And using (2.1), we obtain a system of equations

$$\left.\begin{array}{l} x_1' = x_2 \\ x_2' = x_3 \\ \dots \\ x_{n-1}' = x_n \\ x_n' = f(t, x_1, \dots, x_n) \end{array}\right\} \tag{2.5}$$

Main Results

Now consider the Duffing-type equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = g(t) \tag{3.1}$$

Let $g(t) = 0$, then equation (3.1) becomes

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = 0 \tag{3.2}$$

We obtain the first order systems from the scalar differential equation (3.2) by letting

$$\left.\begin{array}{l} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = w \end{array}\right\} \tag{3.3}$$

The equation (3.2) becomes

$$\left.\begin{aligned} \ddot{y} + a\dot{y} + by + cx + dx^2 + 2x^3 = 0 \\ \dot{z} + az + by + cx + dx^2 + 2x^3 = 0 \\ \dot{z} = -cx - by - az - dx^2 - 2x^3 \end{aligned}\right\} \qquad (3.4)$$

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -cx - by - az - dx^2 - 2x^3 \end{aligned}\right\} \qquad (3.5)$$

The equation (3.5) is the first of the equivalent system obtained directly from the scalar equation. This can be written in matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \quad \Rightarrow \quad \dot{z} = A\underline{z} + g(\underline{z})$$

where the matrix,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}, \quad \underline{\dot{z}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \underline{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } g(\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix}$$

The equation (3.2) can be written as;

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 \equiv \frac{d}{dt}(\ddot{x} + a\dot{x}) + b\dot{x} + cx + dx^2 + 2x^3 = 0 \qquad (3.6)$$

Let $\ddot{x} + a\dot{x} = z$

$$\dot{y} + ay = z \Rightarrow \dot{y} = -ay + z$$

Also (3.6) can be written as;

$$\left.\begin{aligned} \frac{d}{dx}(\ddot{x} + a\dot{x}) + b\dot{x} + cx + dx^2 + 2x^3 \equiv \frac{d}{dt}(z) + b\dot{x} + cx + dx^2 + 2x^3 = 0 \\ \dot{z} + by + cx + dx^2 + 2x^3 = 0 \\ \dot{z} = -by - cx - dx^2 - 2x^3 \end{aligned}\right\} \qquad (3.7)$$

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay + z \\ \dot{z} &= -cx - by - dx^2 - 2x^3 \end{aligned}\right\} \qquad (3.8)$$

The equation (3.8) is the second of the three equivalent first order systems which can be written in matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \Rightarrow \underline{\dot{z}} = A\underline{z} + g(\underline{z})$$

where,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix}, \ \underline{\dot{z}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \underline{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ and \ g(\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix}$$

The equation (3.2) can be written as;

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 \equiv \dddot{x} + \frac{d}{dt}(a\dot{x} + bx) + cx + dx^2 + 2x^3 = 0 \tag{3.9}$$

Let $\quad a\dot{x} + bx = y$

$$ay + bx = \dot{x} \Rightarrow \dot{x} = bx + ay$$

Also equation (3.9) can be written as;

$$\left.\begin{aligned} \dddot{x} + \frac{d}{dt}(a\dot{x} + bx) + cx + dx^2 + 2x^3 \equiv \dddot{x} + \frac{d}{dt}(y) + cx + dx^2 + 2x^3 = 0 \\ \ddot{y} + \dot{y} + cx + dx^2 + 2x^3 = 0 \\ \dot{z} + z + cx + dx^2 + 2x^3 = 0 \\ \dot{z} = -cx - z - dx^2 - 2x^3 \end{aligned}\right\} \tag{3.10}$$

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= bx + ay \\ \dot{y} &= z \\ \dot{z} &= -cx - z - dx^2 - 2x^3 \end{aligned}\right\} \tag{3.11}$$

Equation (3.11) is the third of the equivalent first order system which can be written in a matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \Rightarrow \dot{z} = A\underline{z} + g(\underline{z})$$

where,

$$A = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}, \ \underline{\dot{z}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \underline{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ and \ g(\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix}$$

Stability Analysis

We linearize the nonlinear parts of each of the equivalent system derived using Maclaurin series expansion i.e.

$$g(\underline{x}) = g(\underline{0}) + \underline{x}g'(\underline{0}) + \frac{1}{2!} \parallel \underline{x} \parallel^2 g''(\underline{0}) + \cdots, \qquad \parallel g(\underline{x}) \parallel = 0 \; as \; x \to 0$$

The linearized term is now

$$g(\underline{x}) = g(\underline{0}) + \underline{x}g'(\underline{0})$$

But $g(\underline{0}) = 0$

$$\Rightarrow g(\underline{x}) = \underline{x}g'(\underline{0})$$

Hence, the system $\dot{z} = A\underline{z} + g(\underline{z})$ becomes

$$\dot{x} = A\underline{x} + \underline{x}g'(\underline{0}) \qquad 4.1$$

From the above we see that $g'(\underline{0})$ is necessarily an $n \times n$ matrix since it must be compatible with matrix A. Then we have a linearized system given by $\dot{x} = Bx$, where B is called the linearized matrix i.e $B = A + g'(\underline{0})$.

Applying this we have that

$$g(\underline{x}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \quad \Rightarrow \quad g'(\underline{0}) = \begin{pmatrix} 0 \\ 0 \\ -2d(0) - 6(0)^2 \end{pmatrix} = 0$$

Hence, $\dot{z} = Az$

Now, we want to test the stability for each of the matrices A derived using eigenvalue method i.e. $|A_n - \lambda \mathrm{I}| = 0; n = 1, 2, 3.$

Where, $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -c & -d & -a \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}$

For $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}$,

$$\Rightarrow |A_1 - \lambda \mathrm{I}| = \left| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -c & -b & -a - \lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ -b & -a - \lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -c & -a - \lambda \end{vmatrix} = 0 \quad \Rightarrow -\lambda[-\lambda(-a\lambda) + b] - (0 + c) = 0$$

$$-\lambda^2 a - \lambda^3 - \lambda b + c = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 a + \lambda b + c = 0 \qquad 4.2$$

We have one condition on three constants, two of which are therefore a free choice, choose $a = 1 \; and \; b = c = -1$ for convenience.

Equation (4.2) becomes

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0 \qquad 4.3$$

$$\Rightarrow \quad \lambda = \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}$$

Since all the eigenvalues are not negative real parts then, equation (4.3) is unstable.

For $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix}$,

$$\Rightarrow |A_2 - \lambda \mathrm{I}| = \left| \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -a-\lambda & 1 \\ -c & -b & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda \begin{vmatrix} -a-\lambda & 1 \\ -b & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow -\lambda[-\lambda(-a-\lambda) + b] = 0$$

Putting $a = 1$ *and* $b = c = -1$ gives

$$\lambda^3 + \lambda^2 - \lambda = 0 \tag{4.4}$$

$$\Rightarrow \quad \lambda = \begin{pmatrix} 0 \\ 1.62 \\ -0.62 \end{pmatrix}$$

Since all the eigenvalues are not negative real parts, the equation (4.4) is unstable

For $\quad A_3 = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}$,

$$\Rightarrow |A_3 - \lambda \mathrm{I}| = \left| \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} b-\lambda & a & 0 \\ 0 & -\lambda & 1 \\ -c & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(b-\lambda) \begin{vmatrix} -\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} - a \begin{vmatrix} 0 & 1 \\ -c & -1-\lambda \end{vmatrix} = 0 \qquad \Rightarrow (b-\lambda)(\lambda + \lambda^2) = 0$$

Putting $a = 1$ *and* $b = c = -1$ gives

$$\lambda^3 + 2\lambda^2 + \lambda - 1 = 0 \tag{4.5}$$

$$\Rightarrow \quad \lambda = \begin{pmatrix} 0.46557 \\ -1.232786 + 0.79285 \\ -1.232786 - 0.79285 \end{pmatrix}$$

Equation (4.5) is also unstable.

Conclusion

The eigenvalue method is very easy to handle as verified in this paper. The need to first convert higher order differential equations to first order differential equations has been a key to easily solving higher order differential equations. Therefore, it is recommended that one can check for the stability of the system by picking only one of the equivalent first order systems derived instead of checking for each of the equivalent system. Furthermore, this method tends to be complex if higher numbers are assumed hence; the technique requires the smallest possible number to be assumed. However, since all the eigenvalues of the matrices do not all have negative real parts, we conclude that the Duffing system (3.1) is unstable.

References

Osisiogu, U.A., Eze, E.O. and Obasi, U.E. (2016): Existence and stability of periodic solutions for a class of second order nonlinear differential equations. *Journal of the Nigerian Association of Mathematical Physics*, Vol. 36, No.2, pp 37-41, 43-48.

Kernal, A. (2001): Stability of solution of linear differential equations with periodic coefficient, *Sib. Mat. J.* 41, Number 6, 1005-1010.

Kreici, P. (1974): Hard implicit function theorem and small periodic solutions to partial differential equations. *Commentationes Mathematical Universitatis Carolinae* Vol. 25, No. 3, 519536.

Deepmala, H. and Pathak, K. (2013): A study on some problems on existence of solutions for nonlinear functional-integral equation. *Acta Mathematica Scientia*, 33B(5),pp 1305-1313.

Ueda, Y. (1979): Randomly transitional phenomena in the system, governed by Duffing's equation. *Journal of Statistical Physics*, 20(2), pp. 181-196.

Oyesanya, M.O. and Nwamba, J.I. (2013): Stability analysis of damped cubic-quintic Duffing oscillator. *World Journal of Mechanics* 3: 43-57.

Oyesanya, M.O. and Ejikeme, C.L. (2016). Stability analysis of fractional Duffing oscillator II. *Transactions of the Nigerian Association of Mathematical Physics*. 2, 343-352.

Lakshmanan, S., Rihan, F. A., Rakkiyappan, R. and Park, J. H. (2014): Stability analysis of the differential genetic regulatory networks model with time-varying delays and Markovian jumping parameters, *Nonlinear Analysis: Hybrid Systems*, vol. 14, pp. 1–15.

Sinha, A.S.C. On stability of some third and fourth order delay-differential equations. *Journal of Information and Control*. 23; 165-172.

Sadek, A. I. (2004): On the stability of solutions of certain fourth order delaydifferential equations. *Applied Mathematics and Computation*; 148(2):587–597.

Tunc, C. (2005): Some stability results for the solutions of certain fourth order delay differential equation. *Journal of Differential Equations and Applications*. 4: 165-174.

Bereketoglu, H. (1998). Asymptotic stability in a fourth order delay differential equation. *Journal of Dynam systems Application*. 7(1); 105-115.

Abacus (Mathematics Science Series)
Vol. 50, No 1, April, 2023

HYBRID INERTIAL ALGORITHM FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS FOR BREGMAN RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

Lawal Umar[1], Tafida M. Kabir[2] and Ayuba M. Umar[3]
[1,2]Department of Mathematics, Federal College of Education, Zaria, Nigeria.
[3]Division of Agricultural Colleges. Ahmadu Bello University, Zaria. Nigeria.
Corresponding author: mktafida.555@gmail.com, +2348065539821

Abstract

In this paper, we introduce a modified hybrid inertial iterative algorithm for approximating a common solution of generalized mixed equilibrium problems and fixed points problems for finite family of continuous Bregman relatively nonexpansive mappings in Banach Spaces. Then we prove strong convergence of the sequence to some element in the mentioned set. Our results extend and improve recent results announced by many authors.

1. Introduction

Let E be a real Banach space with norm $\|.\|$, E^* be the dual space of E and let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctions, where $\mathbb{R}$ is the set of real numbers, let $\psi : C \to E^*$ be nonlinear continuous monotone mapping and $\varphi : C \to \mathbb{R}$ be a convex and lower semi continuous function. The generalized mixed equilibrium problem (Darvish 2016) is to find $x \in C$ such that:

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \psi x, y - x \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of generalized mixed equilibrium problem (1.1) is denoted by

$$GMEP(\Theta,\psi,\varphi) = \{x \in C : \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \psi x, y - x \rangle \geq 0, \forall y \in C\}.$$

The equilibrium problems are closely related with other general problems in nonlinear analysis such as fixed point problem, game theory, variational inequality problem and optimization problems. Numerous problems in optimizations, Physics and Economics can be reduced to finding solution of some equilibrium problems. Moreover, various methods have been studied for solutions of some equilibrium problems in Hilbert spaces (see (Blum and Oettli 1994, Combeltes and Hirstoaga 2005, Takahashi and Zembayashi 2009, Zhang and Cho 2016) and the references therein).

In 1964, an inertial algorithm was first proposed and introduced by Polyak (1964) as an acceleration process in solving a smooth convex minimisaion problem. An Inertial-types algorithm is a two-step iterative method in which the next iteration is defined by making use of the previous two iterates. Also an inertial-type algorithm plays a crucial role in speeding up the convergence of the sequence generated by the algorithm. With regards to this importance, a number of researchers have been working on an inertial-type method (see, example (Bot et.al 2015, Bot and Csetnek 2016, Chidume et.al 2018, Dong et.al 2018, Lorenz and Pock 2015) and references therein).

Bregman (1967) introduced an effective technique through Bregman distance function D_f for designing and analyzing feasibility and optimization algorithms. This opened a new area of research in which Bregman's technique is applied in various ways to iterative algorithm for solving not only feasibility problem and optimization problems, but also algorithms for solving fixed point problems for nonlinear mappings (see, example (Ali and Harbau 2016, Chang et.al 2013, Martin-Marquez et.al 2013, Ugwunnadi et.al 2014) and the reference therein).

The normalized duality mapping on E is a mapping $J: E \to 2^{E^*}$ defined by

$$J(x) = \left\{x^* \in E^* : \left\langle x, x^* \right\rangle = \| x \|^2 = \| x^* \|^2 \right\}, \forall x \in E,$$

where $\left\langle x, x^* \right\rangle$ is the pairing between element of E and that of E^*.

Let E be a reflexive Banach space, assume that $f: E \to (-\infty, +\infty]$ is a proper lower semi-continuous and convex function. We denote by $\operatorname{dom} f := \left\{x \in E : f(x) < +\infty\right\}$, the domain of f. Let $x \in \operatorname{int}(domf)$; the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \left\{x^* \in E^* : f(x) + \left\langle x^*, y - x \right\rangle \leq f(y), \forall y \in E\right\}.$$

The Fenchel conjugate of f is the function $f^*: E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\left\{\left\langle x^*, x \right\rangle - f(x) : x \in E\right\}.$$

We know that the Young-Fenchel inequality holds:

$$\left\langle x^*, x \right\rangle \leq f(x) + f^*(x^*), \forall x \in E, x^* \in E^*.$$

A function f on E is coercive (Hiriat and Lemarchal 1993) if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty$$

A function f on E is said to be strongly coercive (Zelinescu 2002) if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\| x \|} = +\infty$$

For any $x \in \operatorname{int} domf$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$$

The function f is said to be Gateaux differentiable at x if $\lim_{t\to 0}\dfrac{f(x+ty)-f(x)}{t}$ exist for any y. In this case, $f^{0}(x,y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. The function f is said to be Gateaux differentiable if it is Gateaux differentiable for any $x\in \operatorname{int}(domf)$. The function f is said to be Frechet differentiable at x if this limit is attained uniformly in y with $\| y \|=1$. Also f is said to be uniformly Frechet differentiable on a subset C of E if the limit is attained uniformly for $x\in C$ and $\| y \|=1$. It is well known that if f is Gateaux differentiable (resp. Frechet differentiable) on int (dom f), then f is continuous and its Gateaux derivative ∇f is norm-to-weak* continuous (resp. norm-to-norm continuous) on int-dom (f), (see (Asplund and Rockafellar 1969, Bonnans and Shapiro 2000)).

Definition 1.1 Bauschke et.al (2001) the function f is said to be:

(i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom f;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.2. If E is a reflexive Banach space, then we have the following results:

(i) f is essentially smooth if and only if f^{*} is essentially strictly convex (Bauschke et.al (2001), Theorem 5.4).

(ii) $(\partial f)^{-1} = \partial f^{*}$ (See Bonnans and Shapiro 2000).

(iii) f is Legendre if and only if f^{*} is Legendre (see Bauschke et.al (2001), Corollary 5.5).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^{*})^{-1}$, ran $\nabla f = dom\nabla(f^{*}) = \operatorname{int}(domf^{*})$ and ran $\nabla f^{*} = domf = \operatorname{int}(domf)$ (see Bauschke et.al (2001), Theorem 5.10), where ran stands for the range.

Examples of Legendre function were given in (Bauschke and Borwein 1997, Bauschke et.al (2001). One important and interesting Legendre function is $\frac{1}{p}\|.\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space; in particular Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty,+\infty]$ is Legendre.

Let $f : E \to (-\infty,+\infty]$ be a convex and Gateaux differentiable function. The function $D_f : dom \times \operatorname{int} domf \to [-\infty,+\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the Bregman distance with respect to f. Also the definition of D_f has the following important property (Reich and Sabah 2011):

$$D_f(z,x) := D_f(z,y) + D_f(y,x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle.$$

Definition 1.3. Let $T : C \to \operatorname{int}(domf)$ with $F(T) \neq \phi$ be a mapping and let $F(T)$ denote the set of fixed points of T, that is., $F(T) = \{x \in C : Tx = x\}$. Then

(1) A point $p \in C$ is said to be an asymptotic fixed point of T if C contain a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Then set of asymptotic fixed points of T is denoted by $\hat{F}(T)$,

(2) quasi-Bregman nonexpansive with respect to f if,

$$F(T) \neq \phi \text{ and } D_f(p,Tx) \leq D_f(p,x), \forall x \in C, p \in F(T);$$

(3) Bregman relatively non expansive with respect to f if,

$$\hat{F}(T) = F(T) \text{ and } D_f(p,Tx) \leq D_f(p,x), \forall x \in C, p \in F(T);$$

(4) Bregman strongly nonexpansive with respect to f and $\hat{F}(T)$ if;

$$F(T) \neq \phi \text{ and } D_f(p,Tx) \leq D_f(p,x), \forall x \in C, p \in \hat{F}(T);$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty}(D_f(p,x_n) - D_f(p,Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(x_n,Tx_n) = 0$$

(5) Bregman firmly nonexpansive (BFNE) with respect to f if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle,$$

equivalently,

$$\begin{aligned} &D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \leq D_f(Tx,y) \\ &+ D_f(Ty,x). \end{aligned} \tag{1.2}$$

Definition 1.4. A mapping $T : C \to C$ is said to be closed, if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y (y \in C)$, then $y = Tx$.

Several results for fixed point approximations of Bregman nonexpansive mappings and their generalizations are established see for example (Ali and Harbau 2016, Chang et.al 2013, Darvish et.al 2019, Kazmi et.al 2018, Ali et.al 2014).

Agha *et.al* (2017) introduced an iterative process which converges strongly to a common element of the sets of solutions of finite family of generalized equilibrium problems, sets of fixed points of finite family of continuous relatively nonexpansive mappings and the sets of finite family of γ-inverse strongly monotone mappings in Banach space as follows. Let the sequences $\{x_n\}$, be generated by the following algorithm:

$$\begin{cases} x_0 \in C_0 = C, chosenarbitrary \\ z_n = \prod_C J^{-1}(Jx_n - \lambda_n A_{n+1} x_n); \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JTx_{n+1} z_n), \\ u_n \in C \ni f_1(u_n, y) + \langle B_1 y_n, y - u_n \rangle \\ + \dfrac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C \\ v_n \in C \ni f_2(\upsilon_n, y) + \langle B_2 y_n, y - \upsilon_n \rangle \\ + \dfrac{1}{r_n} \langle y - \upsilon_n, J\upsilon_n - Jy_n \rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta Ju_n + (1-\beta) J\upsilon_n), \\ C_{n+1} = \{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \prod c_{n+1}(x_0), \forall_n \geq 0, \end{cases}$$

Then, the sequence $\{x_n\}$ converges to some element of F.

Recently, Alansari et.al (2020) studied the following inertial iterative algorithm for variational inequality problem, generalized equilibrium problem and fixed point problem in Banach space. Let the sequence $\{x_n\}$ and $\{z_n\}$ be generated by the algorithm:

$$
\begin{cases}
x_0 = x_1, z_0 \in C, C_0 := C; \\
w_n = x_n + \theta_n (x_n - x_{n-1}) \\
y_n = \prod_C J^{-1}(Jw_n - \mu_n Dw_n); \\
u_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n) JTy_n); \\
z_{n+1} = Tr_n u_n, \\
C_n = \{z \in C : \phi\langle z, z_{n+1}\rangle \le \alpha_n \phi(z, z_n) + (1-\alpha_n)\phi(z, w_n)\}; \\
Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0\rangle \le 0\}; \\
x_{n+1} = \prod C_n \cap Q_n, \forall n \ge 0,
\end{cases}
$$

where $\{\alpha_n\} \subset [0,1], r_n \subset [a,\infty)$ for some $a > 0, \{\theta_n\} \subset (0,1)$ and $\{\mu_n\} \subset (0,\infty)$.

Then sequences $\{x_n\}$, converges strongly to a point $\hat{x} \in \Gamma$.

Very recently, Jantakan and Kaewcharoen (2021) proposed a new iterative method for solving the mixed equilibrium problems and fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach space.

$$
\begin{cases}
x_1 \in C, T_i x_1 = z_1^i \in C; \\
u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)); \\
z_{n+1}^i = \operatorname{Res}_{G,\varphi}^f (u_n^i); \\
C_n^i = \{z \in C : \phi(z, z_{n+1}^i) \le \alpha_n \phi(z, z_1^i) + (1-\alpha_n)\phi(z, x_n)\}, \\
C_n = \cap_{i=1}^N C_n^i; \\
Q_n = \{z \in C : \langle \nabla f(x_1) - f(x_n), z - x_n\rangle \le 0\}, \\
x_{n+1} = proj_{C_n \cap Q_n}^f x_1, \forall n \ge 1,
\end{cases}
$$

where $\{\alpha_n\}$ is a sequence in [0,1] such that $\lim_{n\to\infty} \alpha_n = 0$. Then, the sequence $\{x_n\}$ converges strongly to $proj_\Omega^f x_1$, where $proj_\Omega^f x_1$ is the projection of C onto Ω.

In this paper, motivated and inspired by the results of Agha et.al (2017), Alansari et.al (2020) and Jantakan and Kaewcharoen (2021) mentioned above, we study a modified inertial algorithm for approximating a common solution of generalized mixed equilibrium problems and fixed points problem for Bregman relatively nonexpansive mappings in Banach Spaces. Our results extend and improve recent results announced by many authors.

2. PRELIMINARIES

In this section, we shall consider some Bregman projection and results which will be used in the proof of our main result.

Let E be a reflexive Banach space and $f : E \to (-\infty,+\infty]$ be a Gateaux differentiable and convex function. The Bregman projection of $x \in \operatorname{int} domf$ onto a nonempty, closed and convex set $C \subset domf$ is the necessarily unique vector $proj_C^f(x)$ satisfying

$$D_f(proj_C^f(x), x) := \inf\{D_f(y,x) : y \in C\}.$$

The modulus of total convexity of f at $x \in \operatorname{int} domf$ is the function $\upsilon_f(x,t) : [0,+\infty) \to [0,+\infty)$ defined by

$$\upsilon_f(x,t) := \inf\{Df(y,x) : y \in domf, \|y - x\| = t\}.$$

The function f is called totally convex at x, if $\upsilon_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex, if it is totally convex at any point $x \in \operatorname{int} domf$ and it is said to be totally convex on bounded sets, if $\upsilon_f(B,t) > 0$, for any nonempty bounded subset B of E and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function $\upsilon_f : \operatorname{int} domf \times [0,+\infty) \to [0,+\infty)$ defined by

$$\upsilon_f(B,t) := \inf\{\upsilon_f(x,t) : x \in B \cap domf\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets by Butnariu and Resmerita (2006).

Lemma 2.1 Butnariu and Resmerita (2006). Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function and let $x \in E$. Then:

(i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \le 0, \forall y \in C$;

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \le D_f(y,x), \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Following Alber (1996) and Censor and Lent (1981), let the function $V_f : E \times E^* \to [0,+\infty)$ associated with f is defined by

$$V_f(x,x^*) = f(x)f^*(x^*) - \langle x, x^* \rangle, \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and the following assertions hold:

(1) $V_f(x,x^*) = D_f(x,\nabla f^*(x^*))$ for all $x \in E$ and $y^* \in E^*$

(2) $V_f(x,x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x,x^* + y^*)$ for all $x \in E$ and $y^* \in E^*$.

Lemma 2.2. Naraghired and Yao (2013). Let E be a Banach space $f : E \to \mathbb{R}$ be a Gateaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ be bounded sequences in E. Then

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \text{ if and only if } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. Butnariu and Resmerita (2006). If $x \in domf$, then the following statements are equivalent:

(i) The function f is totally convex at x,

(ii) for any sequence $\{y_n\} \subset domf$,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Recall that the function f is called sequentially consistent Butnariu and Resmerita (2006), if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 2.4. Butnariu and Lusem (2000). Let $f : E \to \mathbb{R}$ be a convex function whose domain contains at least two points. Then f is sequentially consistent if and only if it is totally convex on bounded sets.

Lemma 2.5. Reich and Sabah (2010). Let $f : E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.6. Kazmi et.al (2018). Let $f : E \to \mathbb{R}$ be a Legendre function and C be a nonempty, closed and convex subset of $\operatorname{int} domf$. Let $T : C \to C$ be Bregman quasi nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.

Lemma 2.7. Reich and Sabah (2010). Let $f: E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function, $x_0 \in E$ and C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belong to C. If $D_f(x_n, x_0) \leq D_f(P_C^f x_0, x_0)$ for any $n \in N$, then $\{x_n\}$ strongly converges to $P_C^f x_0$.

The following two results are well known; (see (Zalinescu 2002))

Theorem 2.8. Let E be a reflexive Banach space and let $f: E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:

(1). f is strongly coercive and uniformly convex on bounded subsets of E;

(2). $dom f^* = E^*, f^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets

of E^*

(3). $dom f^* = E^*, f^*$ is Frechet differentiable and ∇f^* is norm-to-norm uniformly continuous on

bounded subsets of E^*.

Theorem 2.9. Let E be a reflexive Banach space and let $f: E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

(1). f is bounded on bounded subsets and uniformly smooth on bounded subsets of E;

(2). f^* is Frechet differentiable and f^* is uniformly norm-to-norm continuous on bounded

subsets of E^* .

(3). $dom f^* = E^*, f^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.

Lemma 2.10. Reich and Sabah (2009). Let $f: E \to \mathbb{R}$ be a uniformly Frechet differentiable and bounded on bounded subsets of E. Then, f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

In order to solve generalized equilibrium problems, we shall consider the following assumptions Blum and Oettli (1994):

The bifunction $\Theta: C \times C \to \mathbb{R}$ satisfies the following assumptions:

(A_1). $\Theta(x, x) = 0, \forall x \in C$;

(A$_2$). Θ is monotone, that is $\Theta(x,y)+\Theta(y,x)\le 0, \forall x,y\in C$;

(A$_3$). For each $x,y,z\in C, \limsup_{t\to 0}\Theta(tz+(1-t)x,y)\le\Theta(x,y)$;

(A$_4$). For each $x\in C, y\to\Theta(x,y)$ is convex and lower semi continuous.

Lemma 2.11. Ali et.al (2019). Let C be a nonempty, closed convex subset of a real reflexive Banach space E. Let $f:E\to\mathbb{R}$ be a convex, continuous and strongly coercive function which bounded on bounded subsets and uniformly convex on bounded subsets of E.

Let $\Theta: C\times C\to\mathbb{R}$ be a bifunction satisfying assumptions (A$_1$) – (A$_4$), let $\psi: C\to E^*$ be a monotone mapping and $\varphi: C\to R$ be a convex and lower semi-continuous function. For $r>0$, a mapping $T_r: E\to C$ is defined by

$$T_r(x)=\left\{z\in C:\tau(z,y)+\frac{1}{r}\langle y-z,\nabla f(z)-\nabla f(x)\rangle\ge 0,\forall y\in C\right\},\forall x\in E$$

where

$$\tau(z,y)=\Theta(z,y)+\langle\psi z,y-z\rangle+\varphi(y)-\varphi(z).$$

Then the following holds:

(a). T_r is single-valued;

(b). T_r is Bregman firmly nonexpansive type mapping (BFNE);

(c). $F(T_r)=GMEP(\Theta,\psi,\varphi)=\hat{F}(T_r)$;

(d). $GMEP(\Theta,\psi,\varphi)$ is closed and convex;

(e). $D_f(p,T_r(w))+D_f(T_r(w),w)\le D_f(p,w),\forall p\in F(T_r), w\in E$.

3. MAIN RESULTS

Theorem 3.1. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f:E\to\mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta_1,\Theta_2: C\times C\to\mathbb{R}$, $k=1,2,$ be bifunctions which satisfying assumptions (A$_1$) – (A$_4$), $\psi_1,\psi_2: C\to E^*$, $k=1,2.$ be continuous monotone mappings and $\varphi_1,\varphi_2: C\to\mathbb{R}$, $k=1,2.$ be convex and lower semi-continuous functions. Let $T_j: E\to E, j=1,2,\dots,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$\Omega := (\cap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)) \cap (\cap_{j=1}^{d} F(T_j)) \neq \phi$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n \rangle + \varphi_1(y) - \varphi_1(u_n) \\ + \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle; \\ v_n \in C \ni \Theta_2(\upsilon_n, y) + \langle \psi_2 y_n, y - \upsilon_n \rangle + \varphi_2(y) - \varphi_2(\upsilon_n) \\ + \frac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \{z \in C_n : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\}; \\ x_{n+1} = proj_{C_{n+1}}^f x_0, \forall n \ge 0, \end{cases}$$

(3.1)

where $\{\alpha_n\},\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty}\alpha_n = 0, \{r_n\} \subset [a,\infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \Theta_k(z,y) + \langle \psi_k x, y - z\rangle + \varphi_k(y) - \varphi_k(z) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall y \in C \right.$$

$\forall x \in E, k = 1,2$.

Then, $\{x_n\}$ converges strongly to $proj_{\Omega}^f x_0$, where $proj_{\Omega}^f x_0$ is the Bregman projection of C onto Ω.

Proof. We divide the proof into a number of steps:

Step 1: We show that $\Omega := (\cap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)) \cap (\cap_{j=1}^{d} F(T_j))$ is closed and convex. It follows from Lemma 2.6 that $\cap_{j=1}^{d} F(T_j)$ is closed and convex. Also from Lemma 2.11(d) that $(\cap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)$ is closed and convex. Therefore

$\Omega := (\cap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)) \cap (\cap_{j=1}^{d} F(T_j))$ is closed convex.

Step 2: We show that C_{n+1} is closed and convex for all $n \geq 0$. Now, by the assumption that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 0$. Let $a, b \in C_{n+1}$ and $z = ta + (1-t)b \in C_{n+1}$, where $t \in [0,1]$ then, we have

$$D_f(a, z_{n+1}) \leq \alpha_n D_f(a, z_n) + (1-\alpha_n) D_f(a, w_n)$$

and

$$D_f(b, z_{n+1}) \leq \alpha_n D_f(b, z_n) + (1-\alpha_n) D_f(b, w_n).$$

Recall that $D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle$. Now, using this definition the above two inequalities are equivalent to

$$\alpha_n \langle \nabla f(z_n), a - z_n \rangle + (1-\alpha_n) \langle \nabla f(w_n), a - w_n \rangle - \langle \nabla f(z_{n+1}), a - z_{n+1} \rangle$$
$$\leq f(z_{n+1}) - f(z_n) - (1-\alpha_n) f(w_n) \tag{3.2}$$

and

$$\alpha_n \langle \nabla f(z_n), b - z_n \rangle + (1-\alpha_n) \langle \nabla f(w_n), b - w_n \rangle - \langle \nabla f(z_{n+1}), b - z_{n+1} \rangle$$
$$\leq f(z_{n+1}) - f(z_n) - (1-\alpha_n) f(w_n). \tag{3.3}$$

Multiply t and $(1-t)$ on both sides of (3.2) and (3.3) respectively, we get

$$\alpha_n \langle \nabla f(z_n), ta + (1-t)b - z_n \rangle + (1-\alpha_n) \langle \nabla f(w_n), ta + (1-t)b - w_n \rangle$$
$$- \langle \nabla f(z_{n+1}), ta + (1-t)b - z_{n+1} \rangle$$
$$\leq f(z_{n+1}) - f(z_n) - (1-\alpha_n) f(w_n).$$

The above inequality yields

$$D_f(ta + (1-t)b, z_{n+1}) \leq \alpha_n D_f(ta + (1-t)b, z_n))$$

$+ (1-\alpha_n) D_f(ta + (1-t)b, w_n) \Rightarrow ta + (1-t)b \in C_{n+1}$ and hence C_{n+1} is closed and convex for all $n \geq 0$. Therefore, C_{n+1} is closed and convex subsets of E.

Step 3: we show that $\Omega \subset C_n$ for all $n \geq 0$, From the assumption that $C_0 = C$, we see that $\Omega \subset C_0 = C$. Suppose that $\Omega \subset C_n$ for all $n \geq 0$, since $T_j : C \to C, j = 1,2,3,...,d$ is a finite family of continuous Bregman relatively nonexpansive mapping. Now for $q \in \Omega \subset C_n$, we obtain the following estimations;

$$D_f(q, z_{n+1}) = D_f(q, \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n) \nabla f(\upsilon_n)))$$
$$\leq \beta_n D_f(q, u_n) + (1-\beta_n) D_f(q, \upsilon_n)$$

$$
\begin{aligned}
&= \beta_n D_f(q, T_{1,r_n}, y_n) + (1-\beta_n) D_f(q, T_{2,r_n}, y_n) \\
&\le D_f(q, u_n) \\
&= D_f(q, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n))) \\
&\le \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, T_j w_n) \\
&\le \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, w_n)
\end{aligned}
$$

(3.4)

that is $q \in C_{n+1}$. This implies by induction that $\Omega \subset C_n$ and the sequence generated (3.1) is well defined for all $n \ge 0$.

Step 4: We show that the sequences $\{x_n\},\{w_n\},\{z_n\}$ and $\{y_n\}$ are bounded. Since $x_n = proj^f_{C_n} x_0$ and $C_{n+1} \subset C_n$ for all $n \ge 0$ by Lemma 2.1, we obtain

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) \le D_f(x_{n+1}, x_0)$$

Implies that

$$D_f(x_n, x_0) \le D_f(x_{n+1}, x_0)$$

This shows that $\{D_f(x_n, x_0)\}$ is non-decreasing. Let $q \in \Omega$. It follows from Lemma 2.1 that

$$D_f(q, proj^f_{C_n} x_0) + D_f(proj^f_{C_n} x_0, x_0) \le D_f(q, x_0)$$

and so

$$D_f(x_n, x_0) \le D_f(q, x_0) - D_f(q, x_n) \le D_f(q, x_0), \forall n \ge 0.$$

Therefore, $\{D_f(x_n, x_0)\}$ is bounded. Consequently $\{D_f(x_n, x_0)\}$ is convergent. It follows that from Lemma 2.5 that $\{x_n\}$ is bounded. Furthermore, the inequality

$$D_f(q, x_n) = D_f(q, proj^f_{C_{n-1} \cap Q_{n-1}} x_0) \le D_f(q, x_0) - D_f(x_n, x_0),$$

this implies that $\{D_f(q, x_n)\}$ is bounded. Now by using the fact that $D_f(q, Tx_n) \le D_f(q, x_n), \forall q \in \Omega$, then $\{Tx_n\}$ is also bounded. Therefore $\{w_n\}$ and $\{y_n\}$ are also bounded. Setting $M = \max\{D_f(q, z_0), \sup_n D_f(q, w_n)\}$. Then obviously $D_f(q, z_0) \le M$. Let $D_f(q, z_n) \le M$ for some n, then it follows from (3.4) that

$$D_f(q, z_{n+1}) \le \alpha_n M + (1-\alpha_n) M \le M$$

Thus, $\{D_f(q, z_{n+1})\}$ is bounded which implies that $\{z_n\}$ is bounded.

By using Lemma 2.1, we obtain

$$D_f(x_m, x_n) = D_f(x_{m,} proj_{C_n}^f x_0) \le D_f(x_m, x_0) - D_f(x_n, x_0)$$

which gives

$$\lim_{n\to\infty} D_f(x_m, x_n) = 0,$$

and holds uniformly for all m. Since f is totally convex on bounded subsets of E, f is sequentially consistent then, it follows from Lemma 2.4 that

$$\lim_{n\to\infty} \|x_m - x_n\| = 0 \ .$$

This implies that the sequence $\{x_n\}$ is Cauchy. Therefore there exists a point $\hat{x} \in C$ such that $x_n \to \hat{x}$ (as $n \to \infty$). Also by using Lemma 2.1, we get

$$D_f(x_{n+1}, x_n) \le D_f(x_{n+1}, x_0) - D_f(x_n, x_0) .$$

This implies

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0 .$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

(3.5)

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we get

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0.$$

(3.6)

Now, by the definition of w_n from (3.1), we have

$$\nabla f(x_n) - \nabla f(w_n) = \nabla f(x_n) - \nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})) .$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(w_n)\| &= \|\theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))\| \\ &\le \theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\|\end{aligned}$$

Using (3.6), implies that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(w_n)\| = 0.$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0 \qquad (3.7)$$

and so, $w_n \to \hat{x}$ as $n \to \infty$.

This also shows that w_n is bounded. Furthermore

$$\|x_{n+1} - w_n\| \le \|x_{n+1} - x_n\| + \|x_n - w_n\|.$$

Using (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} D_f(x_{n+1}, w_n) = 0. \tag{3.8}$$

From the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n) = \langle \nabla f(z_n) - \nabla f(x_{n+1}), q - x_{n+1} \rangle + D_f(q, z_n) - D_f(q, x_{n+1}).$$

Since f is bounded on bounded subset of E^*, then ∇f is bounded on bounded subsets of E^* and hence it follows from boundedness of $\{x_n\}, \{Tx_n\}$ and $\{z_n\}$ that the sequences $\{\nabla f(x_n)\}, \{\nabla f(Tx_n)\}$ and $\{\nabla f(z_n)\}$ are bounded in E^*, which implies that $\{D_f(x_{n+1}, z_n)\}$ is bounded. Since $x_{n+1} = proj_{C_{n+1}}^f x_0 \in C_{n+1} \subset C_n$, we have

$$D_f(x_{n+1}, z_{n+1}) \le \alpha_n D_f(x_{n+1}, z_n) + (1 - \alpha_n) D_f(x_{n+1}, w_n)$$

Using $\lim_{n\to\infty} \alpha_n = 0$ and (3.8), we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we get

$$\lim_{n\to\infty} \|x_{n+1} - z_{n+1}\| = 0.$$

(3.9)

Also

$$\|x_n - z_{n+1}\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}\|.$$

By (3.5) and (3.9), we get

$$\lim_{n\to\infty} \|x_n - z_{n+1}\| = 0.$$

(3.10)

So $z_{n+1} \to \hat{x}$ as $n \to \infty$.

Also this shows that z_{n+1} is bounded.

Taking into account that

$$\|w_n - z_{n+1}\| \le \|w_n - x_n\| + \|x_n - z_{n+1}\|.$$

Using (3.7) and (3.10), we obtain

$$\lim_{n\to\infty}\|w_n - z_{n+1}\| = 0.$$

(3.11)

Since f is uniformly Frechet differentiable, by using Lemma 2.10

$$\lim_{n\to\infty} | f(w_n) - f(z_{n+1}) | = 0,$$

(3.12)

and so

$$\lim_{n\to\infty}\|\nabla f(w_n) - \nabla f(z_{n+1})\| = 0.$$

(3.13)

By Bregman distance, we estimate as follows:

$$\begin{aligned} D_f(q, w_n) - D_f(q, z_{n+1}) &= f(q) - f(w_n) - \langle \nabla f(w_n), q - w_n \rangle \\ &\quad -\big(f(q) - f(z_{n+1}) - \langle \nabla f(z_{n+1}), q - z_{n+1}\rangle\big) \\ &= f(z_{n+1}) - f(w_n) + \langle \nabla f(z_{n+1}), q - z_{n+1}\rangle - \langle \nabla f(w_n), q - w_n\rangle \\ &= f(z_{n+1}) - f(w_n) + \langle \nabla f(z_{n+1}) - \nabla f(w_n), q - w_n\rangle \\ &\quad + \langle \nabla f(z_{n+1}), w_n - z_{n+1}\rangle \end{aligned}$$

(3.14)

Since $\{\nabla f(w_n)\}$ and $\{\nabla f(z_{n+1})\}$ are bounded, for each $q \in \Omega$. By (3.11), (3.12), (3.13) and (3.14), we obtain

$$\lim_{n\to\infty}(D_f(q, w_n) - D_f(q, z_{n+1})) = 0.$$

(3.15)

On the other hand, for each $q \in \Omega$ and $j = 1,2,\ldots,d$, by Lemma 2.11, we have

$$\begin{aligned} D_f(z_{n+1}, y_n) &\le D_f(q, y_n) - D_f(q, z_{n+1}) \\ &= D_f(q, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n))) - D_f(q, z_{n+1}) \\ &\le \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, T_j w_n) - D_f(q, z_{n+1}) \\ &\le \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, w_n) - D_f(q, z_{n+1}) \end{aligned}$$

$$= \alpha_n (D_f(q,z_n) - D_f(q,w_n)) + D_f(q,w_n) - D_f(q,z_{n+1})$$

(3.16)

Since $\{D_f(q,w_n)\}$ and $\{D_f(q,z_{n+1})\}$ are bounded, $\lim_{n\to\infty}\alpha_n = 0$ using (3.15) in (3.16), we get

$$\lim_{n\to\infty} D_f(z_{n+1}, y_n) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we have

$$\lim_{n\to\infty}\|z_{n+1} - y_n\| = 0,$$

(3.17)

and so $y_n \to \hat{x}$ as $n \to \infty$. This shows that $\{y_n\}$ is bounded.

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty}\|\nabla f(z_{n+1}) - \nabla f(y_n)\| = 0,$$

(3.18)

Taking into account that

$$\|y_n - w_n\| \le \|y_n - z_{n+1}\| + \|z_{n+1} - w_n\|.$$

Using (3.17) and (3.11), we obtain

$$\lim_{n\to\infty}\|y_n - w_n\| = 0.$$

(3.19)

Also, since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty}\|\nabla f(y_n) - \nabla f(w_n)\| = 0.$$

(3.20)

By the definition of y_n from (3.1), we have

$$\begin{aligned}
\|\nabla f(y_n) - \nabla f(w_n)\| &= \|\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n) - \nabla f(w_n)\| \\
&= \|\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(w_n) - \nabla f(w_n) \\
&\quad + (1-\alpha_n)(\nabla f(T_j w_n) - \nabla f(w_n))\| \\
&= \|\alpha_n(\nabla f(z_n) + \nabla f(w_n)) + (1-\alpha_n)(\nabla f(T_j w_n) - \nabla f(w_n))\|
\end{aligned}$$

$$\geq (1-\alpha_n)\left\|\nabla f(T_j w_n)-\nabla f(w_n)\right\|-\alpha_n\left\|\nabla f(z_n)-\nabla f(w_n)\right\|,$$

implies that

$$(1-\alpha_n)\left\|\nabla f(T_j w_n)-\nabla f(w_n)\right\|\leq \alpha_n\left\|\nabla f(z_n)-\nabla f(w_n)\right\|+\left\|\nabla f(y_n)-\nabla f(w_n)\right)$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$ and (3.20) that

$$\lim_{n\to\infty}\left\|\nabla f(T_j w_n)-\nabla f(w_n)\right\| = 0, \forall j = 1,2,3,...,d$$

Since f is norm-to-norm uniformly continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}\left\|T_j w_n - w_n\right\| = 0, \forall j = 1,2,3,...,d$$

(3.21)

Step 5: We show that $\hat{x}\in \Omega$. First, we prove that $\hat{x}\in \bigcap_{j=1}^{d} F(T_j)$. Then it follows from the boundedness of the sequence $\{w_n\}$ and E is reflexive, that there exists a subsequence $\{w_{n_m}\}$ of $\{w_n\}$ such that $w_{n_m} \rightharpoonup \hat{x}$ as $m\to\infty$. Also, it follows from (3.7) that there exists a subsequence $\{w_{n_m}\}$ of $\{w_n\}$ such that $w_{n_m} \rightharpoonup \hat{x}$ as $m\to\infty$. Furthermore $\{w_n\}$ is Cauchy sequence, implies that $w_{n_m} \rightharpoonup \hat{x}\in C$ as $m\to\infty$. By using the fact that $w_{n_m} \rightharpoonup \hat{x}\in C$ as $m\to\infty$ and (3.21), we get

$$\lim_{n\to\infty}\left\|T_j w_{n_m} - w_{n_m}\right\| = 0, \forall j = 1,2,3,...,d$$

(3.22)

Since T_j is a finite family of continuous Bregman relatively nonexpansive, using (3.22), we obtain $\hat{x}\in F(T_j) = \hat{F}(T_j), \forall j = 1,2,3,...,d$. Therefore

$$\hat{x}\in \bigcap_{j=1}^{d} F(T_j)$$

Next, we show that $\hat{x}\in \bigcap_{k=1}^{2} GMEP(\Theta_k,\psi_k,\varphi_k) = F(T_{k,r}), k = 1,2.$ Let $q\in \Omega$. First, we prove that $\left\|u_n - y_n\right\| = 0$. Now from $u_n = T_{1,r_n} y_n$, we have

$$D_f(q,u_n) = D_f(q,T_{1,r_n} y_n)$$
$$\leq D_f(q,y_n)$$

$$= D_f(q, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)))$$

$$\leq \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, T_j w_n)$$

$$\leq \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, w_n)$$

$$= \alpha_n D_f(q, z_n) + D_f(q, w_n) - \alpha_n D_f(q, w_n)$$

$$= \alpha_n (D_f(q, z_n) - D_f(q, w_n)) + D_f(q, w_n),$$

implies that

$$D_f(q, u_n) - D_f(q, w_n) \leq \alpha_n (D_f(q, z_n) - D_f(q, w_n))$$

Since $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\lim_{n\to\infty} (D_f(q, u_n) - D_f(q, w_n)) = 0.$$

It follows that

$$\lim_{n\to\infty} D_f(w_n, u_n) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we get

$$\lim_{n\to\infty} \|w_n - u_n\| = 0.$$

(3.23)

Taking into account that

$$\|u_n - y_n\| \leq \|u_n - w_n\| + \|w_n - y_n\|.$$

Using (3.19) and (3.23), we have

$$\lim_{n\to\infty} \|u_n - y_n\| = 0.$$

(3.24)

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|\nabla(u_n) - \nabla f(y_n)\| = 0.$$

From the assumption $r_n \subset [a, \infty)$ and $a > 0$, we get

$$\lim_{n\to\infty} \frac{\|\nabla(u_n) - \nabla f(y_n)\|}{r_n} = 0.$$

Also, since $\{y_n\}$ is bounded, there exist a subsequence $\{y_{n_m}\}$ of $\{y_n\}$ such that $y_{n_m} \rightharpoonup \hat{x} \in C$ as $m \to \infty$. It follows from (3.24) that there exists a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that

$u_{n_m} \rightharpoonup \hat{x} \in C$ as $m \to \infty$. Now

$$\tau(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle \geq 0, \forall y \in C,$$

where

$$\tau(u_n, y) = \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n\rangle + \varphi_1(y) - \varphi_1(u_n).$$

Replacing n by n_m and using (A_2), we get

$$\frac{1}{r_n}\langle y - u_{n_m}, \nabla f(u_{n_m}) - \nabla f(y_{n_m})\rangle \geq -\tau(u_{n_m}, y) \geq \tau(y, u_{n_m})$$

Letting $m \to \infty$, we obtain from $u_{n_m} \rightharpoonup \hat{x} \in C$ that

$$\tau(y, \hat{x}) \leq 0, \forall y \in C,$$

for t with $0 < t \leq 1$ and $y \in C$. Let $yt = ty + (1-t)\hat{x}$, since $y \in C$ and $\hat{x} \in C$, we have

$y_t \in C$ and $\tau(y_t, \hat{x}), y \in C$. Now from ($A_1$) and ($A_3$), we have

$$\begin{aligned} 0 &= \tau(y_t, y_t) \\ &\leq t\tau(y_t, y_t) + (1-t)\tau(y_t, \hat{x}) \\ &\leq t\tau(y_t, y). \end{aligned}$$

Dividing by t, we get

$$\tau(y_t, y) \geq 0, \forall y \in C.$$

Letting $t \to 0$ and using (A_3), we have

$$\tau(\hat{x}, y) \geq 0, \forall y \in C.$$

This implies that $\hat{x} \in GMEP(\Theta_1, \psi_1, \varphi_1)$. Similarly from $u_n = T_{2,r_n} y_n$, by the same argument,

we get $\hat{x} \in GMEP(\Theta_2, \psi_2, \varphi_2)$. Therefore $\hat{x} \in \bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)$.

Hence

$$\hat{x} \in \Omega = (\hat{x} \in \bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)) \bigcap (\bigcap_{j=1}^{d} F(T_j)).$$

Step 6: We show that $x_n \to \hat{x} = proj_{\Omega}^{f} x_0$. Let $\hat{u} = proj_{\Omega}^{f} x_0$. Since $\{x_n\}$ is weekly

convergence, then it follows from $x_{n+1} = proj^{f} C_{n+1} x_0$ and $\hat{u} \in \Omega \subset C_{n+1}$ that

$$D_f(x_{n+1},x_0) \le D_f(\hat{u},x_0).$$

Then by Lemma 2.7, we have $x_n \to \hat{u}$ as $n \to \infty$, thus $\hat{x} = \hat{u}$. Hence the sequence $\{x_n\}$ converges strongly to a point $\hat{u} = proj_{\Omega}^{f} x_0$. Therefore, it follows from uniqueness of the limit that $\{x_n\}$ converges strongly to a point $\hat{x} = proj_{\Omega}^{f} x_0$. This completes the proof. ■

From Theorem (3.1), if $\varphi_1 = \varphi_2 = 0$, we have the following corollary;

Corollary 3.2. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f: E \to \mathbb{R}$ be coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R}$, $k = 1,2.$ be bi functions which satisfying assumptions (A_1) – (A_2), $\psi_1, \psi_2 : C \to E^*$, $k = 1,2.$ be continuous monotone mappings. Let $T_j : E \to E, j = 1,2,3,...,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} GEP(\Theta_k, \psi_k)\right) \bigcap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*\big(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))\big); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n \rangle + \dfrac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n) \rangle; \\ v_n \in C \ni \Theta_2(\upsilon_n, y) + \langle \psi_2 y_n, y - \upsilon_n \rangle + \dfrac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n) \rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \{z \in C : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\}; \\ x_{n+1} = proj_{C_{n+1}}^{f} x_0, \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \Theta_k(z,y) + \langle \psi_k x, y - z \rangle + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x) \rangle \ge 0, \forall y \in C \right\}, \forall x \in E, k = 1,2$$

Then, $\{x_n\}$ converges strongly to $proj_{\Omega}^{f} x_0$, where $proj_{\Omega}^{f} x_0$ is the Bregman projection of C onto Ω.

From theorem (3.1), if $\psi_1 = \psi_2 = 0$ and $\varphi_1 = \varphi_2 = 0$, we obtain the following corollary;

Corollary 3.3. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ be coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R},\ k = 1,2.$ be bi functions which satisfying assumptions (A_1) – (A_4). Let $T_j : E \to E, j = 1,2,3,...,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} EP(\Theta_k)\right) \bigcap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*\left(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))\right); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \Theta_1(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle; \\ v_n \in C \ni \Theta_2(\upsilon_n, y) + \dfrac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \left\{z \in C : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\right\}; \\ x_{n+1} = proj_{C_{n+1}}^{f} x_0, \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \Theta_k(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall y \in C \right\}, \forall x \in E, k = 1,2.$$

Then, $\{x_n\}$ converges strongly to $proj_{\Omega}^{f} x_0$, where $proj_{\Omega}^{f} x_0$ is the Bregman projection of C onto Ω.

From theorem (3.1), if $\Theta_1 = \Theta_2 = 0$ and $\varphi_1 = \varphi_2 = 0$, we have the following corollary;

Corollary 3.4. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\psi_1, \psi_2 : C \to E^*$ be continuous monotone mappings. Let $T_j : E \to E, j = 1,2,3,\ldots,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} VIP(\psi_k, C)\right) \bigcap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \langle \psi_1 y_n, y - u_n \rangle + \dfrac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle; \\ v_n \in C \ni \langle \psi_1 y_n, y - \upsilon_n \rangle + \dfrac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\}; \\ x_{n+1} = proj^f_{C_{n+1}} x_0, \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \langle \psi_k y_n, y - u_n \rangle + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C \right\}, \forall x \in E, k = 1,2.$$

Then, $\{x_n\}$ converges strongly to $proj^f_{\Omega} x_0$, where $proj^f_{\Omega} x_0$ is the Bregman projection of C onto Ω.

Corollary 3.5. Let C be a nonempty closed and convex subset of a reflexive Banach space E. Let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R}$, $k = 1,2.$ be bi functions which satisfying assumptions (A_1) – (A_4), $\psi_1, \psi_2 : C \to E^*$, $k = 1,2.$ be continuous monotone mappings and $\varphi_1, \varphi_2 : C \to \mathbb{R}$, $k = 1,2.$ be convex and semi-continuous functions. Let $T_j : E \to E, j = 1,2,3,\ldots,d$ be finite family of continuous relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)\right) \bigcap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n)JT_j w_n); \\ u_n \in C \ni \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n \rangle + \varphi_1(y) - \varphi_1(u_n) \\ + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle; \\ \upsilon_n \in C \ni \Theta_2(\upsilon_n, y) + \langle \psi_2 y_n, y - \upsilon_n \rangle + \varphi_2(y) - \varphi_2(\upsilon_n) \\ + \dfrac{1}{r_n}\langle y - \upsilon_n, J\upsilon_n - Jy_n \rangle; \\ z_{n+1} = J^{-1}(\beta_n Ju_n + (1-\beta_n)J\upsilon_n); \\ C_{n+1} = \{z \in C : \phi(z, z_{n+1}) \le \alpha_n \phi(z, z_n) + (1-\alpha_n)\phi(z, w_n)\}; \\ x_{n+1} = \prod_{C_{n+1}} x_0, \forall n \ge 0, \end{cases}$$

where j is normalized duality mapping, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \Theta_k(z, y) + \langle \psi_k x, y - z \rangle + \varphi_k(y) - \varphi_k(z) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \forall y \in C \right\},$$

$\forall x \in E,\ k = 1,2.$

Then, $\{x_n\}$ converges strongly to $\prod_\Omega x_0$ is the projection of C onto Ω.

4. APPLICATIONS

In this section, we present some applications of theorem 3.1 as follows:

4.1. Finite family of continuous Bregman relatively nonexpansive mappings and system of equilibrium problems. By setting $\psi \equiv 0$, $\varphi \equiv 0$ in theorem 3.1, the sequence $\{x_n\}$ defined in 3.1 converges strongly to $proj^f_\Omega x_0$ where $\Omega := \left(\bigcap_{k=1}^2 EP(\Theta_k)\right) \bigcap \left(\bigcap_{j=1}^d F(T_j)\right) \ne \phi$ and $EP(\Theta)$ is the solutions of the equilibrium problem for Θ.

4.2. Finite family of continuous Bregman relatively nonexpansive mappings and system of convex optimization problems. By setting $\Theta \equiv 0$, $\psi \equiv 0$ in theorem 3.1, the sequence $\{x_n\}$ defined in 3.1 converges strongly to $proj^f_\Omega x_0$ where

$\Omega := \left(\bigcap_{k=1}^{2} GMP(\psi_k)\right) \bigcap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi$ and $CMP(\varphi)$ is the solutions of the convex optimization problem for φ.

4.3. Finite family of continuous Bregman relatively nonexpansive mappings and system of variational inequalities problems. By setting $\Theta \equiv 0$, $\varphi \equiv 0$ in theorem 3.1, the sequence $\{x_n\}$ defined in 3.1 converges strongly to $proj_{\Omega}^{f} x_0$ where $\Omega := \left(\bigcap_{k=1}^{2} VIP(\psi_k, C)\right) \bigcap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi$ and $VIP(C, \psi)$ is the solutions of the variational inequalities problem for ψ.

References

Agha, O. I., Ibiam, L. O., Madu, E. U., Ofoedu, C. E., Onyi and H. Zegeye. (2017). Hybrid Algorithm for Nonlinear Equilibrium, Variational Inequality and Fixed Point Problems. *Journal of Physical Research. 7(1); ISSN; 2141 – 8403.*

Alansari, M., Ali, R and Farid, M. (2020). Strong Convergence of an Inertial Iterative Algorithm for Variational Inequality Problem, Generalised Equilibrium Problem and Fixed Point Problem in a Banach Space. *Journal of Inequalities and Applications. 42; 1 - 22.*

Alber, Y. I. (1996). Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications, in: A. G. Kartsatos (Ed). *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Mercel Dekker, New York.*

Ali, B., Ezeora, J. N and Lawan M. S. (2019). Inertial Algorithm for Solving Fixed Point and Generalized Mixed Equilibrium Problems in Banach Space. *Pan American Mathematical Journal. 29(3); 64 – 83.*

Ali, B and Harbau, M. H. (2016). Convergence Theorem for Bregman K-Mappings and Mixed Equilibrium Problems in Reflexive Banach Spaces. *Journal of Function Space. ID 5161682.*

Ali, B., Ugwunnadi, G. C., Idris, I and Minjibir, M. S. (2014). Strong Convergence Theorem for Quasi Bregman Strictly Pseudocontractive, Mappings and Equilibrium Problems in Banach Spaces. *Fixed Point Theory and Applications. 231; 1 – 17. Doi :10.1186/1687-1812.*

Asplund, E and Rockafellar, R. T. (1969). Gradient of Convex Function, *Trans. Am. Math. Soc. 228; 443 – 467.*

Bauschke, H. H and Borwein, J. M. (1997). Legendre Function and the Method of Bregman Projections. *Journal of Convex Analysis. 4; 27 – 67.*

Bauschke, H. H., Combettes, P. L and Borwein, J. M. (2001). Essential Smoothness , Essential Strict Convexity and Legendre Functions in Banach Spaces. *Commun. Contemp. Math. 3; 615 – 647.*

Bonnans, J. F and Shapiro, A. (2000). Perturbation Analysis of Optimization Problems. *Springer New York.*

Bot, R. I., Csetnek, E. R and Hendrich, C. (2015). Inertial Douglas-Racheord Splitting for Monotone Inclusion Problems. *Appl. Math. Comput. 256; 472 – 478.*

Bregman, L. M. (1967). The Relaxation Method for Finding the Common Point of Convex Set and its Application to Solution of Convex Programming. *USSR Comput. Math. Phys.7; 200 – 217.*

Butnariu, D and Lusem, A. N. (2000). Totally Convex Function for Fixed Points Computation and Infinite Dimensional Optimization. *Kluwer Academic Dordrecht.*

Butnariu, D and Resmerita, E. (2006). Bregman Distance, Totally Convex Functions and a Method for Solving Operator Equations in Banach Spaces. *Abstr. Anal. Article ID:84919.*

Blum, E and Oettli, W. (1994). From Optimization and Variational Inequalities to Equilibrium Problems. *Math. Stud. 63; 123 – 145.*

Bot, R. I and Csetnek, E. R. (2016). An Inertial Forward-Backward-Forward Primal-Dual Splitting Algorithm for solving Monotone Inclusions Problems. *Numer Algorithms. 71; 519 – 540.*

Censor, Y and Lent, A. (1981). An Iterative Row-Action Method for Interval Convex Programming. *Journal of Optimization Theory and Application. 34; 321 – 353.*

Chang, S. S., Wang, L., Wang, X, R and Chan, C. K. (2013). Strong Convergence Theorems for Bregman Totally Quasi-asymptotical Nonexpansive Mappings in Reflexive Banach Spaces. *Appl. Math. Comput. Doi:101016/jamc2013,11.074.*

Chidume, C. E., Ikechukwu, S. I and Adamu, A. (2018). Inertial Algorithm for Approximating a Common Fixed Point for Countable Family of Relatively Nonexpansive Mappings. *Fixed Point Theory and Applications. 9; 1 – 9.*

Combeltes, P. L and Hirstoaga, S. A. (2005). Equilibrium Programming in Hilbert Spaces. *Nonlinear Convex Analysis. 6; 117 – 136.*

Darvish, V. (2016). Strong Convergence Theorem for a System of Generalized Mixed Equilibrium Problems and Finite Family of Bregman Nonexpansive Mappings in Banach Spaces. *Opsearch. 53; 584 – 603. DOI10.1007/s12597 – 015 – 1245 – 2.*

Darvish, V., Jantakawa, A. Kaewcharoen, A and Biranvand, N. (2019). Strong Convergence Theorem for Solving a Generalized Mixed Equilibrium Problems and Finding Fixed Point of a Weak Bregman Relatively Nonexpansive Mappings in Banach Spaces. Functional Analysis arXiv: 1911.02246(Math), Cornell University.

Dong, O. L., Yuan, H. B., Je, C. Y and Rassias, T. M. (2018). Modified Inertial Algorithm and Inertial CQ Algorithm for Nonexpansive mappings. *Optim. Lett. 12; 87 – 102.*

Hiriart-Urruty and Lemarchal, J. B. (1993). Convex Analysis and Minimization Algorithm. Grudlehren der Mathematischen Wissenchaften, 306, Springer, Berlin.

Jantakan, K and Kaewchroen, A. (2021). Strong Convergence Theorems for Mixed Equilibrium Problems and Bregman Relatively Nonexpansive Mappings in Reflexive Banach Spaces with Application. *Journal Nonlinear Science and Application. 14; 63 – 79.*

Kazmi, K. R., Ali, R and Yousuf, S. (2018). Generalized Equilibrium and Fixed point Problems for Bregman Relatively Nonexpansive Mappings in Banach Spaces. *Journal of Fixed Point Theory and Applications. 20; 151.* [https://doi.org/10.1007/s11784 - 018-0627](https://doi.org/10.1007/s11784-018-0627)*-1.*

Lorenz, D and Pock, T. (2015). An Inertial Forward-Backward Algorithm for Monotone Inclusions. *Journal of Mathematics Imaging. 51; 311 – 325.*

Martin-Marquez, Reich, V and Sabach, S. (2013). Iterative Method for Approximating Fixed Points of Bregman Nonexpansive Operators. *Discrete Contin, Dyn. Syst. 6; 1043 – 1063.*

Naraghired, E and Yao, J. C. (2013). Bregman Weak Relatively Nonexpansive Mappings in Banach Spaces. *Fixed Point Theory and Applications. 1 - 43. Doi:10.1186/1687 – 1812 – 2013 – 141.*

Polyak, B. T. (1964). Some Methods of Speeding up the Convergence of Iteration Methods. *USSR Comput. Math. Phys. 4(5); 1 – 7.*

Reich, S and Sabah, S. (2009). A Strong Convergence Theorem for a Proximal-Type Algorithm in reflexive Banach Spaces. *Journal of Nonlinear Convex Analysis. 10; 471 – 485.*

Reich, S and Sabah, S. (2010). Two Strong Convergence Theorems for a Proximal Method in Reflexive Banach spaces. *Numer. Funct. Anal. Optim. 31; 22 – 44.*

Reich, S and Sabah, S. (2011). A Projection Method for Solving Nonlinear Problems in Banach spaces. *Journal of Fixed Point Theory and Application. 9; 101 – 116.*

Takahashi, W and Zembayashi, K. (2009). Strong and Weak Convergence Theorems for Equilibrium Problems and relatively Nonexpansive Mappings in Banach Spaces. *Nonlinear Analysis. 70; 45 – 57.*

Zalinescu, C. (2002). Convex Analysis in General vector Spaces. World Scientific, River Edge.

Zhang, M and Cho, S. Y. (2016). A Monotone Projection Algorithm for Solving Fixed Points of Nonlinear Mappings and Equilibrium Problems. *Journal of Nonlinear Sciences and Applications. 9; 1453 – 1562.*